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# RIGID FINITE ELEMENT AND LIMIT ANALYSIS*

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**ABSTRACT**: According to the lower-bound theorem of limit analysis the Rigid Finite Element Method (RFEM) is applied to structural limit analysis and the linear programmings for limit analysis are deduced in this paper. Moreover, the Thermo-Parameter Method (TPM) and Parametric Variational principles (PVP) are used to reduce the computational effort while maintaining the accuracy of solutions. A better solution is also obtained in this paper.

**KEY WORDS**: rigid finite element method, Limit analysis, plastic theory, thermo-parametric method, parametric variational principles

## I. INTRODUCTION

Finite Element Method(FEM) is a well-known powerful method in the structural analysis, but the computational amount is very large when using linear FEM incrementally to deal with a nonlinear problem. So it is necessary to study a special approach for each special problem according to its characteristics.

In some plastic problems such as the Prandtl solution of the punch problem[1] a structure will become a number of pieces which can slide with respect to each other when it is loaded to its limit state. The elastic deformation can be ignored in the state of plastic flow, so these pieces can be considered approximately as rigid bodies. In addition the limit solutions are exactly the same for both the ideal rigid plastic model and the ideal elastic plastic model. As a conclusion of the above discussion, it is reasonable and feasible to adopt RFEM[2] in structural limit analysis.

RFEM gives a better solution for nonlinear problem and its computational effort is also less than that of FEM[3,4].

For these reasons some basic ideas of RFEM, such as considering elements as rigid bodies in their limit state, are adopted and the linear programmings for limit analysis are deduced from the lower bound theorem by adopting RFEM. In addition TPM[5] and PVP[6] are used to further reduce the computational effort and to improve the accuracy of solution. In this way a better solution is obtained and a large scale problem can be solved with micro-computer such as IBM PC/AT.

## II. BASIC ASSUMPTIONS AND RIGID FINITE ELEMENT METHOD

According to the traditional limit analysis theory the following basic assumptions are used

(1) Ideal rigid plastic assumption: material is considered as rigid before yield but elements moved rigidly with respect to each other after yield.

(2) Rigid movement assumption: every element is assumed as a rigid body, so the displacement $\boldsymbol{u}$ of point $P(x, y)$ within element can be completely described by the rigid movement $(u_g, v_g, \theta)$ of the corresponding centroid $G(x_g, y_g)$. That is

$$\boldsymbol{u}=\begin{bmatrix}1 & 0 & y-y_g\\ 0 & 1 & x_g-x\end{bmatrix}\begin{Bmatrix}u_g\\ v_g\\ \theta\end{Bmatrix} \tag{1}$$

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(3) Strain and stress field: the relative displacement $\boldsymbol{\delta}_j=[\delta_d\ \delta_s]^T$ of a point on element seams shown in Fig.1 can be expressed as[2] (shown in Fig. 2)

$$\boldsymbol{\delta}_j=B_j U^j_G \tag{2}$$

where

$$B=\begin{bmatrix} -l_1 & -m_1 & m_1(x-x_{g1})-l_1(y-y_{g1}) & l_1 & m_1 & l_1(y-y_{g2})-m_1(x-x_{g2}) \\ -l_2 & -m_2 & m_2(x-x_{g1})-l_2(y-y_{g1}) & l_2 & m_2 & l_2(y-y_{g2})-m_2(x-x_{g2}) \end{bmatrix}$$

$$U^j_G=[u_{g1}\quad v_{g1}\quad \theta_1\quad u_{g2}\quad v_{g2}\quad \theta_2]^T$$

$l_1$, $m_1$, $l_2$, $m_2$ are the normal and tangential direction cosines of the element seam $\beta_j$ .Eq.(3) is the strain field of RFEM.

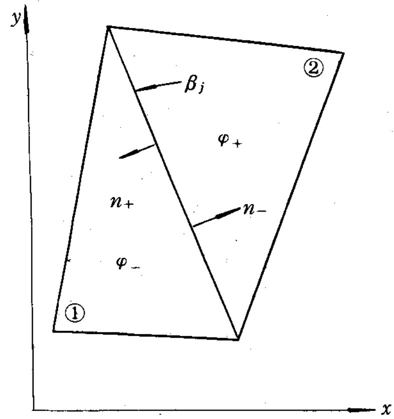



Fig. 1 Rigid elements

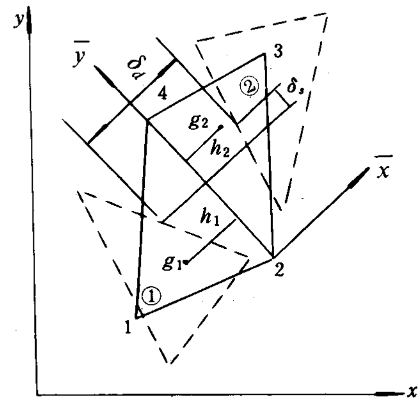



Fig. 2 Relative displacement between elements

The normal stress $\sigma_n$ and tangential stress $\tau_s$ on element seam $\beta_j$ are taken as the stress field $\boldsymbol{R}_j$ of RFEM, such as

$$\boldsymbol{R}_j=[\sigma_n\ \ \tau_s]^T \tag{3}$$

The relation between stress field $\boldsymbol{R}_j$ and strain field $\boldsymbol{\delta}$ is given in Ref. [2], namely

$$\boldsymbol{R}_j=D\boldsymbol{\delta}_j \tag{4}$$

where $D$ is the elastic matrix, which is

$$D=\frac{1}{h_1+h_2}\begin{bmatrix} \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & \dfrac{E}{2(1+\nu)} \end{bmatrix} \tag{5}$$

in plane strain problem, in which $h_1$ and $h_2$ are the perpendicular distance from centroids of two adjacent rigid elements to their contact line.

Dividing a structure into a number of arbitrary convex polygon elements and analyzing the forces of every element, we obtain the global equilibrium equation as

$$K\boldsymbol{U}=\boldsymbol{P} \tag{6}$$

in which

$$K=\sum_j\int_{\beta_j}B^TDB\mathrm{d}s$$

$$\boldsymbol{U}=[u_{g1}\ \ v_{g1}\ \cdots\ u_{gN}\ \ v_{gN}]^T$$

$\boldsymbol{P}$ is the external load vector.

The rigid displacement $\boldsymbol{U}$ of every element's centroid under the external load $\boldsymbol{P}$ can be deduced from Eq. (6) and then the displacement of any point within elements can be obtained from Eq. (1).

## III. LIMIT ANALYSIS BASED ON THE LOWER BOUND THEOREM

The lower-bound theorem points out that the limit load corresponding to any statical admissible stress field, which satisfies the equilibrium equation with the force boundary condition and doesn't violate the yield condition, isn't greater than the real limit load. In this paper the normal stress $\sigma_n$ and tangential stress $\tau_s$ are taken to establish the statical admissible stress field.

Statical admissible stress field $\boldsymbol{R}_j$ should satisfy the equilibrium condition with the force boundary condition, so the stress field $\boldsymbol{R}_j$ must satisfy the following virtual displacement equation if it equilibrates with the external load $\boldsymbol{P}$ in an arbitrary virtual displacement field $\boldsymbol{u}$.

$$\iiint_V \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma}\, \mathrm{d}V + \sum_j \int_{\beta j} \boldsymbol{\delta}_j^{\mathrm{T}} \boldsymbol{R}_j\, \mathrm{d}s - \lambda \iiint_V \boldsymbol{u}^{\mathrm{T}} \boldsymbol{P}\, \mathrm{d}V = 0 \tag{7}$$

where $\boldsymbol{\varepsilon}$ is the strain within element and $\boldsymbol{\sigma}$ is the stress tensor of element. Because elements ar rigid bodies, the strain $\boldsymbol{\varepsilon}$ within elements is equal to zero, so that the first item of Eq. (7) is equal to zero too. The substitution of Eq. (2) into Eq. (7) gives

$$-\sum_j \boldsymbol{U}_G^{j\mathrm{T}}\, B_j^{\mathrm{T}} \boldsymbol{R}_j l_j + \lambda \sum_e \boldsymbol{U}_e^{\mathrm{T}} \boldsymbol{P}_e = 0 \tag{8}$$

in which $\sum_j$ means summing up over all element s eams, $\sum_e$ means summing up over all elements and $l_j$ is the length of the element seam $\beta_j$ . Writing Eq. (8) in vector form and considering the arbitrarity of virtual displacement vector we will get

$$-\boldsymbol{B}^{\mathrm{T}} \boldsymbol{R} + \lambda \boldsymbol{P} = 0 \tag{9}$$

where $B$ is formed by $l_j B_j$ coming from element seam $\beta_j$, $\boldsymbol{R}$ is formed by $\boldsymbol{R}_j$ from element seam $\beta_j$ and $\boldsymbol{P}$ is the external load vector formed by $\boldsymbol{P}_e$ from element $e$.

The linear programming for limit analysis can be deduced from the lower bound theorem and Eq. (9). That is

Find $\boldsymbol{R} = ?$

max $\lambda(\boldsymbol{R})$

$$\text{s. t.} \begin{cases} -B^{\mathrm{T}} \boldsymbol{R} + \lambda \boldsymbol{P} = 0 \\ f \leqslant 0 \\ \lambda \geqslant 0 \end{cases} \tag{10}$$

Eq. (10) is a nonlinear mathematical programming for a nonlinear problem such as Mises yield condition, so it will take a large amount of computational time. For the convenience of solving, the yield condition is linearized[4]. That is

$$f(\boldsymbol{R}) = N\boldsymbol{R} - \boldsymbol{K}_s \leqslant 0 \tag{11}$$

in which $N = \mathrm{diag}(N_j)$, $\boldsymbol{K}_s = [K_1, K_2, \cdots, K_j, \cdots,]^{\mathrm{T}}$ . The coefficients N and $\boldsymbol{K}_s$ in Eq. (11) will take different forms for different yield conditions. Mises condition, which represents a ellipse region in $\sigma_n - \tau_s$ space, can be approximated by six lines. Such as

$$N_j = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & \sqrt{2}/2 & -\sqrt{2}/2 & -1 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^{\mathrm{T}} \tag{12}$$

$$K_j = [\,0.5 \quad \sqrt{2}/2 \quad \sqrt{2}/2 \quad 0.5 \quad \sqrt{2}/2 \quad \sqrt{2}/2\,]^{\mathrm{T}} \sigma_0$$

In view of the nontension condition of soil material the Coulomb condition, which is composed of two lines, will lead to

$$N_j=\begin{bmatrix} \mathrm{tg}\,\varphi & 1 \\ \mathrm{tg}\varphi & -1 \\ 1 & 0 \end{bmatrix}$$

$$K_j=[\ C\ C\ 0]^{\mathrm{T}} \tag{13}$$

so that the linear programming for limit analysis is expressed as

$$\begin{aligned} &\text{Find } \boldsymbol{R}=?\\ &\max\ \ \lambda(\boldsymbol{R})\\ &\text{s. t.} \begin{cases} -B^{\mathrm{T}}\boldsymbol{R}+\lambda\boldsymbol{P}=0\\ N\boldsymbol{R}\leqslant\boldsymbol{K}_s\\ \lambda\geqslant 0\end{cases} \end{aligned} \tag{14}$$

where $\boldsymbol{R}$ is a free variable. Thus Eq. (14) is a linear programming with free variables and has been discussed in Ref. [7].

The total number of constraints is greater than that of variables in Eq. (14). To reduce the computational effort, the following dual linear programming should be solved instead of Eq. (14)

$$\begin{aligned} &\text{Find } \dot{U},\ \dot{\boldsymbol{\beta}}\\ &\min\ \ \lambda=\boldsymbol{K}_s^{\mathrm{T}}\ \dot{\boldsymbol{\beta}}\\ &\text{s.t.} \begin{cases} B\dot{U}-N^{\mathrm{T}}\dot{\boldsymbol{\beta}}=0\\ \dot{U}^{\mathrm{T}}\boldsymbol{P}=1\\ \dot{\boldsymbol{\beta}}\geqslant 0\end{cases} \end{aligned} \tag{15}$$

Eq. (15) is also a linear programming with free variables and can be solved using the algorithm proposed in Ref. [7].

## IV. THERMO-PARAMETRIC METHOD[5]

The well-known Melan's theory for plastic shakedown analysis states that if such a self-equilibrating residual stress field $\boldsymbol{R}_0$ whose composite stress field with the elastic stress $\boldsymbol{R}_E$ corresponding to the complex loading satisfies the yield condition $f(\boldsymbol{R}_E+\boldsymbol{R}_0)\leqslant 0$ can be found, then the structure will be shakedown. The structural limit analysis can be treated similarly as the shakedown analysis by considering it as a special case of shakedown analysis. The key to the problem is how to find a time independent self-equilibrating residual stress field. In TPM a thermo stress field is selected as the self-equilibrating residual stress field and then the temperature parameters are adjusted to maximize the load multiplier factor under the condition of satisfying the yield condition (11). That is

$$\begin{aligned} &\text{Fin T}=?\\ &\max\ \lambda(\mathrm{T})\\ &\text{s.t.} \begin{cases} \lambda \mathrm{N}\boldsymbol{R}_E+\mathrm{N}\,\boldsymbol{R}_0(\mathrm{T})\leqslant\boldsymbol{K}_s\\ \lambda\geqslant 0\end{cases} \end{aligned} \tag{16}$$

The self-equilibrating residual stress field $\boldsymbol{R}_0$ which uses temperature parameters as variables has been set up for RFEM in Ref. [4]. Such as

$$\boldsymbol{R}^j_0=(DBK^{-1}G^j-H^j)\boldsymbol{T}^j \tag{17}$$

where

$$G^j=\begin{bmatrix} G^j_{11} & G^j_{21} & G^j_{31} & G^j_{41} & G^j_{51} & G^j_{61}\\ G^j_{12} & G^j_{22} & G^j_{32} & G^j_{42} & G^j_{52} & G^j_{62}\end{bmatrix}$$

$$G^j_{i1}=\frac{1}{4}\ LD_{11}\alpha h\int_{-1}^{1}(1-\eta)B_{1i}(\eta)\,d\eta$$

$$G^j_{i2}=\frac{1}{4}\ LD_{11}\alpha h\int_{-1}^{1}(1+\eta)B_{1i}(\eta)\,d\eta$$

$$H^j=\begin{bmatrix} H_{11} & H_{12} \\ 0 & 0 \end{bmatrix}$$

$$H_{11}=\frac{1}{2}D_{11}\alpha h(1-\eta) \qquad H_{12}=\frac{1}{2}D_{11}\alpha h(1+\eta)$$

$$\boldsymbol{T}^j=[T_1\ T_2]^j$$

$$i=1,\cdots,6;\ j=1,\cdots,NJ$$

$\alpha$, the coefficient of heat expansion which has no effect on the computational results, is only used to establish the self-equilibrating residual stress field and has nothing to do with the real material being used, so it can be taken as 1.0. $NJ$ is the total number of element seams, $T_1$ and $T_2$ are the temperature parameters of two ends of an elasto-plastic seam.

Summing up Eq. (17) for all element seams we can obtain the self-equilibrating residual stress field $\boldsymbol{R}_0$. That is

$$\boldsymbol{R}_0=(DBK^{-1}G-H)\boldsymbol{T} \tag{18}$$

It is easy to verify that the residual stress field $\boldsymbol{R}_0$ satisfies the self-equilibrating condition. Introducing Eq. (18) into Eq. (16) we can get the linear programming for limit analysis, such as

$$\begin{aligned}&\text{Find } \boldsymbol{T}=?\\ &\max\ \lambda(\boldsymbol{T})\\ &\text{s. t.}\begin{cases}\lambda N\boldsymbol{R}_E+N(DBK^{-1}G-H)\boldsymbol{T}\leqslant \boldsymbol{K}_s\\ \lambda\geqslant 0\end{cases}\end{aligned} \tag{19}$$

In Eq. (19) the total number of variables and constraints are all smaller than those of Eqs. (14) and (15), so the computational effort should be less than that of Eqs. (14) and (15). But the accuracy of Eq. (19) is no better than that of Eqs. (14) and (15) because it uses the nodal temperature parameters as variables to set up the self-equilibrating residual stress field instead of adjusting the stress field directly.

## V. PARAMETRIC VARIATIONAL PRINCIPLES

The structure's plastic limits can also be obtained from the elasto-plastic analysis, but the computational amount of the elasto-plastic analysis with the incremental method is too large to be used in some complex structures. PVP is proposed recently to solve a problem with indefinite boundary in mathematical physics. Iterations are avoided in each incremental step and a great step can be adopted as well, so PVP can reduce the computational effort significantly and will expand the problem scale for limit analysis using the elasto-plastic analysis method. The quadratic programming for elasto-plastic analysis was deduced in Ref. [3] for RFEM. That is

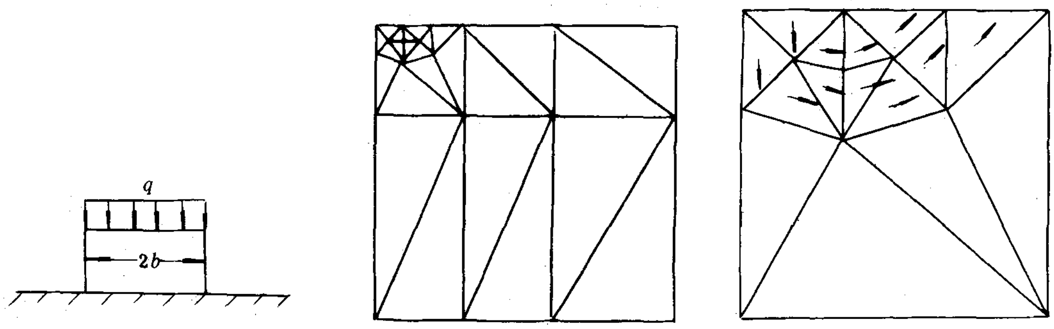
$$\begin{aligned}&\text{Min } \Pi=\frac{1}{2}\,\mathrm{d}\,\boldsymbol{U}^{\mathrm{T}}K\,\mathrm{d}\,\boldsymbol{U}-\mathrm{d}\,\boldsymbol{U}(\varphi\boldsymbol{\lambda}+\boldsymbol{q})\\ &\text{c.b.}\begin{cases}C\,\mathrm{d}\,\boldsymbol{U}-M\boldsymbol{\lambda}-\boldsymbol{d}+\boldsymbol{\nu}=0\\ \boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{\lambda}=0,\ \boldsymbol{\nu}\geqslant 0,\ \boldsymbol{\lambda}\geqslant 0\end{cases}\end{aligned} \tag{20}$$

where $K$ is the assembled stiffness matrix of structures, $dU$ is the incremental displacement vector, $\boldsymbol{q}$ is the incremental load vector, and $\boldsymbol{\lambda}$ is the plastic flow factor vector. The matrices $\varphi$, $C$ and $M$ are concerned with the plastic potential and yield function respectively.

The above three methods for limit analysis have different advantages. For example, the accuracy of Eqs. (14), (15) and (20) are better than that of Eq. (19), but their computational amount is also larger. How to choose these methods depends on the demand for accuracy and computational amount. For a complex structure Eq. (19) should be chosen to reduce the computational work significantly while losing a little accuracy.

## VI. NUMERICAL EXAMPLES

1. A frictionless punch problem shown in Fig. 3 (a) can be solved by sliding line theory. The solution for Hill's and Prandtl's sliding line field is $q/2k=2.571$ and the upper bound solution is $q/2k=2.89$. This problem is analyzed here by using Mises yield condition and the results are tabled in Tab. 1. The sliding model obtained here is shown in Fig. 3(c) as well. These results are very close to the solutions of sliding line theory.




(a) Punch problem (b) Finite element mesh (c) Sliding model

Fig. 3

**Table 1**
The results of Mises condition

| h/b | Eq. (14) | Eq. (19) | Eq. 20 | Sliding solution [1] |
|---|---|---|---|---|
| 1 | 1.0 | 1.0 | 1.0 | 1.0 |
| 10 | 2.677 | 2.643 | 2.677 | 2.571 |

2. A weightless slope with angle 135 ° under a uniform pressure $q$ is analyzed by using Coulomb condition and the results are tabled in Tab. 2.Here the sliding line field presented in Ref. [1] is adopted as finite element mesh, so the results obtained here are very close to the sliding solution. In general it is difficult to coincide the finite element mesh with sliding line field in advance, but an arbitrary dense mesh can also be used to find a good limit solution as shown in the next example.

**Table 2**
Results comparison

| Eq. (14) | Eq. (19) | Eq. (20) | sliding solution |
|---|---|---|---|
| 3.5763 | 3.483 | 3.5775 | 3.5708 |

3. A frictionless strip footing shown in Fig. 5 is analyzed using Coulomb condition with cohesion $C=10\text{kpa/m}^2$ and internal friction angle $\varphi=20°$. If the finite element mesh is adopted in accordance with the sliding line filed, an excellent solution would be obtained. To illustrate the generality of these methods, an arbitrary mesh shown in Fig.5 is used and its results are tabled in Tab. 3. It can be found that the results obtained here are close to the upper bound. This is caused by adopting the rigid assumption which restricts the collapse model, so the limit solution will be greater than the exact one.

**Table 3**
Results comparison

| Eq. (14) | Eq. (19) | Eq. (20) | Lower bound[8] | Upper bound[8] |
|---|---|---|---|---|
| 171.18 | 171.134 | 171.6 | 143.0 | 175.0 |

4. A 90 ° notched tension specimen shown in Fig. 6 is analyzed for Mises yield condition When loading the stress concentration will occur in the notch and it will be destroyed first. The

limit load can be got from the equilibrium equation as

$$P_{limit} = 20 \qquad \sigma_0 = 60.0$$

The limit load obtained here is 60.0 too, so the accuracy of the solutions is excellent.

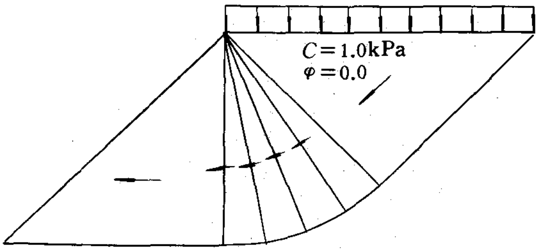



Fig. 4 The limit solution of slope

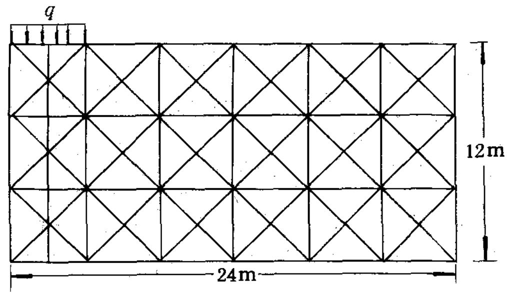



Fig. 5 Frictionless strip footing

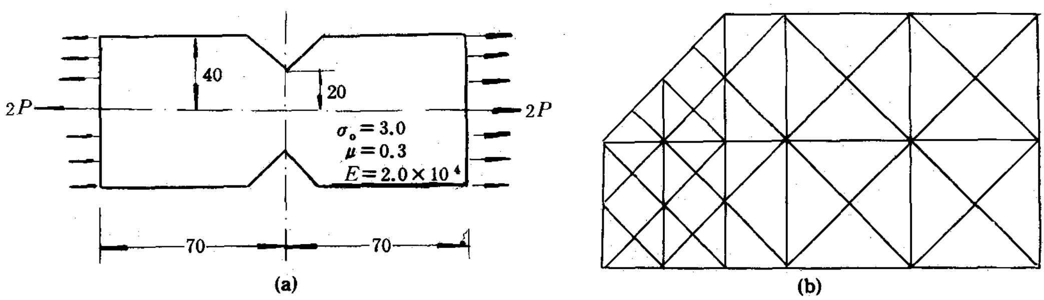



Fig. 6 A 90° notched tension specimen

## VII. CONCLUSION

In this paper three methods for limit analysis are proposed by adopting the ideas of rigid finite element method and the computational effort for limit analysis is significantly reduced. These methods have different advantages and should be chosen according to different demands. It can be found from numerical examples that the limit solutions of Eq. (14) are very close to those of elasto-plastic analysis, so RFEM is an effective and promising method for limit analysis.

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