Block-interfaces model for nonlinear numerical simulations of rock structures*

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ABSTRACT

Based on the real characteristics of rock structures, a block-interfaces model that simulates the nonlinear mechanical behavior of discontinuous structures such as jointed rock masses is presented. In this model, the continuity condition of inter-blocks is relaxed by the Lagrangian multiplier method so that both continuum and discontinuum problems can be solved and the discontinuities of rock structures are fully taken into account. Furthermore, by using the parametric variational principle the elastoplastic analysis and contact analysis of rock structures are deduced to the minimization of the potential energy functional under the constraint of the yield and contact conditions. In this way only a linear complementary problem needs to be solved for contact analysis and elastoplastic analysis. Moreover, the non-associate flow and strain softening problem can be dealt with easily and the iterations used in the traditional elastoplastic and contact analysis methods are avoided so that the computational effort is significantly reduced. Lastly, numerical examples about a jointed rock mass are given to illustrate the accuracy and effectiveness of the model. The proposed model is currently implemented for two-dimensional problems, but it could easily be extended into three-dimensional problems.

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INTRODUCTION

Because of the presence of weak structural surfaces and joints, the behavior of rock structures is largely controlled by the discontinuities. Up to now, many numerical models have been established to simulate the behavior of this kind of structure. These models can be classified into two categories. In the first category, models which can represent the discontinuities are introduced into the finite element method (FEM), such as Linkage Element [1], Joint Element [2], Equivalent Continuum Model [3,4], Damage Mechanics Theory [5]. In the other category, a rock structure is modeled as many rock blocks connected by joints and interfaces, and includes models such as Discrete Element Method (DEM) [6], Block Theory [7], Discontinuous Deformation Analysis [8] and Rigid Finite Element Method [9-15] . In the DEM, engineering problems are modeled as a large system of distinct interacting general shaped bodies and the dynamic contact topology of the bodies is determined by the solution of the equation of motion of every body, so that the DEM is a very computationally intensive procedure. The discontinuities were fully taken into consideration in this category of methods, so that these methods provide an effective way to the numerical simulation of discontinuous structures.

Based on the real characteristics of rock structures, the rigid body elastoplastic interfaces model was proposed by combining the basic ideas of FEM and DEM to simulate the mechanical behavior of rock structures [14], and the variational principles for rigid finite element were also established [15]. In this model rock masses were considered as rigid bodies, as a result the deformation and failure of rock masses were not taken into account. In this paper, the block-interfaces model is presented to simulate the nonlinear mechanical behavior of both rock masses and joints. Here rock masses are simulated by blocks while weak structural surfaces and joints are simulated by interfaces. Not only interfaces but also blocks can have various nonlinear

constitutive relations, such as elastoplastic, visco-plastoelastic, and other relations. The block-interfaces model can be used to simulate the nonlinear behavior of discontinuous structure, especially rock structure, as well as continuous structure.

Relaxing the continuity condition of inter-blocks by the Lagrangian multiplier method, a modified functional for discontinuum is established that the displacement can be discontinuous between blocks. In this way the discontinuities of a discontinuum are very easily treated. Similarly a modified functional for continuum is also established by the penalty function method. Through selecting a suitable penalty function, continuous problems are dealt with in the same way as discontinuous problems. So this model can be used to analyze discontinuous problems as well as continuous problems. The model is reduced to a constant stress element for the continuum, but the element can have arbitrary convex polygon shapes. The block-interfaces model can simulate the mechanical behavior of rock structures, such as slopes and foundations.

The contact behavior between rock masses can be reduced to a non-associate plastic flow problem. In the traditional incremental iteration methods, many iterations must be performed in each increment step to obtain a better and convergent solution. The convergence is very sensitive to the step size and the coefficients matrix for non-associate plastic flow problem is asymmetric. Moreover, the strain softening problems is also difficult to be dealt with. The Parametric Variational Principle (PVP) [16] is an effective method for the boundary value problems with variable boundaries, such as elastoplastic and contact problems. In PVP the elastoplastic and contact problems are transformed to the minimization of a functional under the constraints of the yield and contact conditions so only a linear complementary problem is solved. This method does not require iterations in each incremental step and only requires a few Gaussian eliminations. The convergence is not sensitive to the step size (a too large step size will lead to a worse solution, but

it is still convergent) and so a better efficiency and precision can be achieved. Furthermore, the non-associate plastic flow and strain softening problems can also be treated straightforwardly.

DEFORMATION PATTERN OF BLOCK-INTERFACES MODEL

Rock structures are composed of a large number of rock masses and weak structural surfaces, so that the movement and deformation of rock masses can be simulated by blocks while the weak structural surfaces and contact surfaces between rock masses can be simulated by interfaces. Figure 1 illustrates the basic idea of the block-interfaces model.

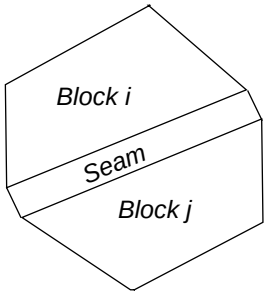

Figure 1 Block-interfaces model

In the rigid finite element method, the elements are rigid bodies and the displacement field is a first order incomplete polynomial [11]. In order to describe the deformation of the blocks, a first order complete polynomial or higher order polynomial should be used. Here the first order complete polynomial is used. Let x_0, y_0 denote the coordinate of the centroid of a block, u_0, v_0 the rigid translation displacement of the block at its centroid, θ_0 the rigid rotation of the block, and $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ the mean strain components of the block. The displacement components *u*, *v* at any point (*x*, *y*) within the block can be obtained by these parameters as (in two dimension):

$$\boldsymbol{u} = \boldsymbol{Q} \cdot \boldsymbol{d} \tag{1}$$

where $\boldsymbol{u} = [u, v]^T$, $\boldsymbol{d} = [u_0, v_0, \theta_0, \varepsilon_x, \varepsilon_y, \gamma_{xy}]^T$

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & y_0 - y & x - x_0 & 0 & (y - y_0)/2 \\ 0 & 1 & x - x_0 & 0 & y - y_0 & (x - x_0)/2 \end{bmatrix}$$

Equation 1 represents a constant strain state. For elastic problem the stress-strain relation is

$$\sigma = \boldsymbol{D}_b \cdot \varepsilon \tag{2}$$

in which D_b is the elasticity matrix of blocks while the strain vector ε and the stress vector σ are

$$\varepsilon = [\varepsilon_x, \varepsilon_y, \gamma_{xy}]^T, \quad \sigma = [\sigma_x, \sigma_y, \tau_{xy}]^T$$

where $\sigma_x, \sigma_y, \tau_{xy}$ are the mean stress components of a block.

Consider the deformation of a point P in a typical interface element of the block-interfaces model (Figure 2). Generally the width *h* of the interface is far less than its length *L*, so that the strain of the interfaces in the *n* direction can be treated as constant. Denoting the relative displacement in the *n* direction between the top and bottom surfaces at point *P* of the interface by δ_n and the relative displacement in the *s* direction by δ_s, the relative displacement vector $\delta = [\delta_n, \delta_s]^T$ can be expressed as

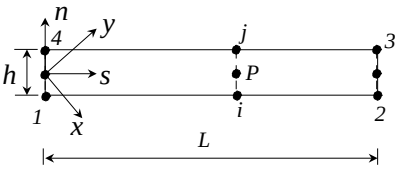

Figure 2 Interface element

$$\delta = \bar{\boldsymbol{u}}_j - \bar{\boldsymbol{u}}_i \tag{3}$$

where $\bar{\boldsymbol{u}}_i$ and $\bar{\boldsymbol{u}}_j$ are the displacement vector at points *i* and *j* (Figure 2) in the local coordinate system *n*-*s*. The relation between the displacement vector $\bar{\boldsymbol{u}}$ in the local coordinate system *n*-*s* and the corresponding displacement vector $\boldsymbol{u}$ in the global coordinate system *x*-*y* is

$$\bar{\boldsymbol{u}}_k = \boldsymbol{T}\boldsymbol{u}_k \quad (k = i, j) \tag{4}$$

where $\boldsymbol{T}$ is a coordinate transforming matrix formed by the direction cosine of *n* and *s*.

The relative displacement vector δ between the top and bottom surfaces of the interfaces can be finally obtained by substituting equation (1) into equation (4) and then into equation (3) as

$$\delta = \boldsymbol{B} \cdot \boldsymbol{X} \tag{5}$$

where $B = [-TQ(x_i, y_i), TQ(x_j, y_j)]$

$$\boldsymbol{X} = [u_0^i, v_0^i, \theta_0^i, \varepsilon_x^i, \varepsilon_y^i, \gamma_{xy}^i, u_0^j, v_0^j, \theta_0^j, \varepsilon_x^j, \varepsilon_y^j, \gamma_{xy}^j]^T$$

Because the width *h* of the interfaces is far less than their length *L*, the normal strain in the *s* direction can be neglected compared with the other strain components. The strain vector in the local coordinate system at any point *P* in the interfaces can be expressed as [14]

$$\varepsilon' = \frac{1}{h}\delta \tag{6}$$

where $\varepsilon' = [\varepsilon_n, \gamma_{ns}]^T$, ε_n is the normal strain of the interface in the *n* direction and γ_{ns} the shear strain of the interface.

The normal stress at the interfaces in the *s* direction can also be neglected compared with other stress components. The relation between the stress vector *R* and the relative displacement vector δ can be obtained from the above equations as

$$\boldsymbol{R} = \boldsymbol{D}_s \cdot \delta \tag{7}$$

where $\boldsymbol{D}_s = \frac{1}{h}\begin{bmatrix} \frac{E}{1-\nu^2} & 0 \\ 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$ (for a plane stress problem)

$\boldsymbol{R} = [\sigma_n \ \ \tau_s]^T$, σ_n is the normal stress of the interface in the *n* direction and τ_s the shear stress.

After assigning various nonlinear constitutive relations to interfaces, the block-interfaces model can be used to simulate the weak structural surfaces between rock masses. The contact surfaces between rock masses can be treated as a special case of interfaces, that is to say, the control equations of contact problems can also be assigned to interfaces as a kind of special

elastoplastic constitutive relation and so the weak structural surfaces of rock structures and the contact surfaces between rock masses can be treated similarly. The mechanical behavior of interfaces is represented by the relative displacement between their adjacent blocks, so it is unnecessary to create interface elements when meshing.

STATIC EQUILIBRIUM EQUATION

The discontinuities of rock structures are fully taken into consideration in the block-interfaces model, but the continuity condition between blocks must be satisfied in the natural variational principles and this is inconvenient to discontinuum mechanics. In order to deal with discontinuum problems, the continuity condition between blocks should be relaxed by the Lagrange multiplier method. That is to say, the continuity condition will be introduced into the energy functional as a constraint condition to establish a modified functional so that the minimization of the original function under constraint of the continuity condition will lead to a stationary problem of the modified functional without constraint conditions. The displacement continuity condition between blocks can be expressed as

$$\delta = 0 \tag{8}$$

The modified functional established by the Lagrange multiplier method is

$$\Pi = \Pi_e + \Pi_f + \Pi_P + \Pi_L \tag{9}$$

where Π_e is the elastic strain energy of blocks, Π_f and Π_P are the potential energy of distributed load and concentrated load respectively, Π_L is the term introduced by the Lagrangian multiplier method. The elastic strain energy of blocks is

$$\Pi_e = \sum_e t \iint_{\Omega_e} \frac{1}{2} \boldsymbol{\varepsilon}^T \boldsymbol{D}_b \boldsymbol{\varepsilon} \, dxdy = \sum_e \frac{1}{2} \boldsymbol{\varepsilon}^T (tS_e \boldsymbol{D}_b) \boldsymbol{\varepsilon} \tag{10}$$

where $\varepsilon = [\varepsilon_x, \varepsilon_y, \gamma_{xy}]^T$ is the mean strain vector of block *e*, *t* is the thickness of the block, S_e is the area of the block and Ω_e represents block *e*.

The potential energy of distributed load $\boldsymbol{f}_e$ is

$$\Pi_f = -\sum_e \iint_{\Omega_e} \boldsymbol{u}^T \boldsymbol{f}_e dxdy = -\sum_e \boldsymbol{d}^T (\iint_{\Omega_e} \boldsymbol{Q}^T \boldsymbol{f}_e dxdy) \tag{11}$$

The potential energy of concentrated force $\boldsymbol{P}_m$ is

$$\Pi_p = -\sum_m \boldsymbol{u}^T \boldsymbol{P}_m = -\sum_m \boldsymbol{d}^T (\boldsymbol{Q}^T \boldsymbol{P}_m) \tag{12}$$

It can be proved that the Lagrange multiplier (defined along interface β_k between blocks) is just the stress vector $\boldsymbol{R}$ composed of normal stress σ_n and tangential stress τ_s of the inter-blocks. In fact, the term introduced by the Lagrange multiplier method is the work done by the stress vector $\boldsymbol{R}$ in the discontinuous displacement of the inter-blocks, namely

$$\Pi_L = \sum_k t \int_{\beta_k} \frac{1}{2} \delta^T \boldsymbol{D}_s \delta ds = \sum_k \frac{1}{2} \boldsymbol{X}^T (t \int_{\beta_k} \boldsymbol{B}^T \boldsymbol{D}_s \boldsymbol{B} ds) \boldsymbol{X} \tag{13}$$

It can be seen that the term Π_L established by the Lagrangian multiplier method is just the strain energy of interfaces [15]. Letting the first order variation of the modified functional Π equal zero, it can be proved that the stationary of the modified functional is equivalent to the minimization of the original functional. The global equilibrium equation can be obtained from the stationary of the modified functional Π as

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} \tag{14}$$

where $\boldsymbol{K} = \sum_e t S_e \boldsymbol{D}_b + \sum_j t \int_{\beta_j} \boldsymbol{B}^T \boldsymbol{D}_s \boldsymbol{B} ds$

$$\boldsymbol{P} = \sum_{e} \iint_{\Omega_e} \boldsymbol{Q}^T \boldsymbol{f}_e dxdy + \sum_{m} \boldsymbol{Q}^T \boldsymbol{P}_m$$

$$\boldsymbol{U} = [u_0^1, v_0^1, \theta_0^1, \varepsilon_x^1, \varepsilon_y^1, \gamma_{xy}^1, u_0^2, v_0^2, \theta_0^2, \varepsilon_x^2, \varepsilon_y^2, \gamma_{xy}^2, \cdots]^T$$

For a continuum, the displacement is continuous between blocks. The continuity condition between elements can be satisfied by the penalty function method. After deriving similarly as the above, it can be found that the continuum mechanics problems can be solved by letting E in equation (7) be a large number. To discuss the property of the solution of the penalty function method, equation(14) can be rewritten as

$$(\boldsymbol{K}_1 + E\boldsymbol{K}_2)\boldsymbol{U} = \boldsymbol{P} \tag{15}$$

where $\boldsymbol{K}_1$ is the first term of the stiffness matrix in Eq(14) and $E\boldsymbol{K}_2$ is the second term. $\boldsymbol{K}_1$ only restrains the mean strain of the blocks and $\boldsymbol{K}_2$ the relative displacement between blocks. So $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are all singular. The solution of Equation (15) is non-trivial when $E \to \infty$ and then the block-interfaces model is deduced to an arbitrary polygon constant strain element.

STATE EQUATION

The following conditions should be satisfied for elastoplastic analysis

$$\text{Stress-strain relation} \quad \begin{cases} d\sigma = \mathbf{D}_b (d\varepsilon - d\varepsilon^p) & \textit{in blocks} \\ dR = \mathbf{D}_s (d\delta - d\delta^p) & \textit{in interfaces} \end{cases} \tag{16}$$

$$\text{Yield condition} \quad \begin{cases} f_a^b(\sigma, \varepsilon^p, k) \le 0 \quad (a = 1,2,\cdots,L_e) & \textit{in blocks} \\ f_a^s(\mathbf{R}, \delta^p, k) \le 0 \quad (a = 1,2,\cdots,L_j) & \textit{in interfaces} \end{cases} \tag{17}$$

$$\text{Plastic flow criteria} \quad \begin{cases} d\varepsilon^p = \lambda_a^b \dfrac{\partial g_a^b}{\partial \sigma} & \textit{in blocks} \\ d\delta^p = \lambda_a^s \dfrac{\partial g_a^s}{\partial \mathbf{R}} & \textit{in interfaces} \end{cases} \tag{18}$$

$$\begin{aligned} \lambda_a^b &\begin{cases} > 0 & \text{if } f_a^b = 0 \\ = 0 & \text{if } f_a^b < 0 \end{cases} \\ \lambda_a^s &\begin{cases} > 0 & \text{if } f_a^s = 0 \\ = 0 & \text{if } f_a^s < 0 \end{cases} \end{aligned} \tag{19}$$

where ε^p is the mean plastic strain of the blocks, f_a^b the yield condition of the blocks, g_a^b the plastic potential function of the blocks, L_e the number of yield functions which should be satisfied in block *e*, λ_a and λ_s the plastic flow parameters and *k* the hardening parameter. δ^p is the plastic relative displacement between the top and bottom surface of interfaces, *dR* the incremental stress vector composed of the normal and tangential incremental stresses of the interfaces and L_j the number of yield conditions which should be satisfied at interface β_j.

Expanding the yield function f_a^b and f_a^s at the current stress state with the first order *Taylor* series, we have

$$\begin{aligned} f_a^b(\sigma, \varepsilon^p, k) &= f_a^{b0} + \left(\frac{\partial f_a^b}{\partial \sigma}\right)^T d\sigma + \left(\frac{\partial f_a^b}{\partial \varepsilon^p}\right) d\varepsilon^p + \left(\frac{\partial f_a^b}{\partial k}\right) dk \\ f_a^s(\mathbf{R}, \delta^p, k) &= f_a^{s0} + \left(\frac{\partial f_a^s}{\partial \mathbf{R}}\right)^T d\mathbf{R} + \left(\frac{\partial f_a^s}{\partial \delta^p}\right) d\delta^p + \left(\frac{\partial f_a^s}{\partial k}\right) dk \end{aligned} \tag{20}$$

in which f_a^{b0} and f_a^{s0} denote the value of yield functions f_a^b and f_a^s at the current stress state respectively and *dk* can be assumed as a linear function of the plastic flow parameters, that is

$$dk = h_i \lambda_i \quad \begin{cases} (i = 1,2,\cdots, L_e) & \textit{for blocks} \\ (i = 1,2,\cdots, L_j) & \textit{for intfaces} \end{cases} \tag{21}$$

After substituting equations (16), (18) and (21) into equation (20), we have the following equation in matrix notation

$$\begin{aligned} f^{b0} + \boldsymbol{W} d\varepsilon - \boldsymbol{M}\lambda^{b} \le 0 \\ f^{s0} + \boldsymbol{Z} d\delta - \boldsymbol{S}\lambda^{s} \le 0 \end{aligned} \tag{22}$$

where the elements of matrix $\boldsymbol{W}$, $\boldsymbol{M}$, $\boldsymbol{Z}$ and $\boldsymbol{S}$ are

$$\begin{aligned} \boldsymbol{W}_a &= \left(\frac{\partial f_a^b}{\partial \sigma}\right)^T \boldsymbol{D}_b \quad (\alpha = 1,2,\cdots,L_e) \\ \boldsymbol{M}_{ai} &= \left(\frac{\partial f_a^b}{\partial \sigma}\right)^T \boldsymbol{D}_b \left(\frac{\partial g_i^b}{\partial \sigma}\right) - \left(\frac{\partial f_a^b}{\partial \varepsilon^p}\right)^T \left(\frac{\partial g_i^b}{\partial \sigma}\right) - \frac{\partial f_a^b}{\partial k} h_i \quad (i = 1,2,\cdots L_e) \\ \boldsymbol{Z}_a &= \left(\frac{\partial f_a^s}{\partial R}\right)^T \boldsymbol{D}_s \quad (\alpha = 1,2,\cdots,L_j) \\ \boldsymbol{S}_{ai} &= \left(\frac{\partial f_a^s}{\partial R}\right)^T \boldsymbol{D}_s \left(\frac{\partial g_i^s}{\partial R}\right) - \left(\frac{\partial f_a^s}{\partial \delta^p}\right)^T \left(\frac{\partial g_i^s}{\partial R}\right) - \frac{\partial f_a^s}{\partial k} h_i \quad (i = 1,2,\cdots L_j) \end{aligned} \tag{23}$$

The matrix $\boldsymbol{M}$ and $\boldsymbol{S}$ in equation (22) is a L_e-th and L_j-th order matrix. Generally $\boldsymbol{M}_{ai}$ and $\boldsymbol{S}_{ai}$ are not equal to zero only when the stress state is located at the intersection point between the two yield surfaces f_a and f_i, so the matrix $\boldsymbol{M}$ and $\boldsymbol{S}$ can be treated approximately as a diagonal matrix to save the computational effort.

The plastic flow factor λ is greater than zero only if the corresponding yield function $f = 0$; otherwise λ equal zero in equation (22). By introducing a slack factor ν which is complementary with λ, equation (22) will become

$$\begin{cases} f^{b0} + \boldsymbol{W} d\varepsilon - \boldsymbol{M}\lambda^{b} + \nu^{b} = 0 \\ f^{s0} + \boldsymbol{Z} d\delta - \boldsymbol{S}\lambda^{s} + \nu^{s} = 0 \\ \nu^{bT}\lambda^{b} = 0,\ \nu^{sT}\lambda^{s} = 0,\ \lambda^{b} \ge 0,\ \ \nu^{b} \ge 0,\ \lambda^{s} \ge 0,\ \ \nu^{s} \ge 0 \end{cases} \tag{24}$$

The above equation is the state equation which should be satisfied in the variational process.

PARAMETRIC VARIATIONAL PRINCIPLE FOR BLOCK-INTERFACES MODEL

The incremental iteration method, the variational inequality method [17] and the Parametric Variational Principle (PVP) [16] are frequently used in elastoplastic and contact problems. In the incremental iteration method, a large number of iterations are necessary and the convergence is also very sensitive to the step size. Moreover, the stiffness matrix for non-associate plastic flow problems is asymmetric. PVP overcomes these disadvantages and can easily deal with the non-associate plastic flow problems. Iterations are avoided and the convergence is not sensitive to the step sizes, so it has a better efficiency and convergence. Starting from the variational inequalities, the variational inequality method also give a very similar formulation to that of PVP for elasto-plastic analysis, so it has the same characteristics as those of PVP.

The basic idea of PVP is that the plastic flow factor is considered as a control parametric variable and is incapable of variation. The energy functional is minimized under the constraint of the yield conditions.

Dividing rock structures into blocks with the block-interfaces model, we have the following virtual work equation in incremental form

$$\begin{aligned}\delta W = \sum_{j=1}^{NJ} t\int_{\beta_j} \delta(d\delta)^T d\boldsymbol{R} ds + \sum_{e=1}^{NE} t\iint_{\Omega_e} \delta(d\varepsilon)^T d\sigma dxdy - \\ \sum_m \delta(du)^T d\boldsymbol{P}_m - \sum_e \iint_{\Omega_e} \delta(d\boldsymbol{u})^T d\boldsymbol{f}_e dxdy = 0\end{aligned} \tag{25}$$

where *NJ* is the total number of interfaces and *NE* is the total number of blocks.

Substituting equations (16) and (18) into equation (25) and considering that the plastic flow factor λ is incapable of variation, we obtain the functional as

$$\Pi^* = \sum_{e=1}^{NE} t\iint_{\Omega_e} \frac{1}{2} d\varepsilon^T \boldsymbol{D}_b d\varepsilon dxdy + \sum_{j=1}^{NJ} t\int_{\beta_j} \frac{1}{2} d\delta^T \boldsymbol{D}_s d\delta ds - \sum_e \iint_{\Omega_e} d\boldsymbol{u}^T d\boldsymbol{f}_e dxdy - \sum_m d\boldsymbol{u}^T d\boldsymbol{P}_m - \sum_{e=1}^{NE_1} t\iint_{\Omega_e} \lambda^T \boldsymbol{Y} d\varepsilon dxdy - \sum_{j=1}^{NJ_1} t\int_{\beta_j} \lambda^T \boldsymbol{V} d\delta ds \tag{26}$$

where NJ_1 is the number of interfaces in the plastic state and is not greater than the total number NJ of the interfaces. NE_1 is the number of blocks in the plastic state and is not greater than the total number NE of blocks. The elements of matrices V and Y are

$$\boldsymbol{V}_a = \left(\frac{\partial g_a^s}{\partial \boldsymbol{R}}\right)^T \boldsymbol{D}_s, \qquad \boldsymbol{Y}_a = \left(\frac{\partial g_a^b}{\partial \sigma}\right)^T \boldsymbol{D}_b \tag{27}$$

In the above discussion, the plastic flow factor λ is a function of other basic state variables (such as stress and strain), but it is a control parametric variable in PVP and so it is incapable of variation. Taking first order variation to Equation (26) and deriving similarly as that in Ref. (10), it can be proved that the minimization of the functional Π^* under the control of the state equation (24) is equivalent to the equilibrium condition, so the incremental displacement field solved here is the real incremental displacement field. The parametric variational principle for the block-interfaces model can be stated as: **After dividing rock structure into *NE* blocks and *NJ* interfaces, in all possible incremental displacement fields, the real field will lead to the global minimization of the functional Π^* under the control of the state function (24)**.

In the block-interfaces model, blocks are in constant stress state and interfaces are in linear stress state. For the sake of convenience, we assume that the yield conditions are satisfied in the mean sense, namely, we should integrate the state functions over the blocks and interfaces. The total number of the state functions is L (L= $\sum_{j=1}^{NJ_1} L_j + \sum_{e=1}^{NE_1} L_e$).

According to the parametric variational principle, elastoplastic analysis by the

block-interfaces model can be deduced to minimization of the functional Π^* under the control of the state function (24). Substituting equations (1) and (3) into the functional Π^* and minimizing the functional Π^* under the control of the state function (24), we obtain the equation for solving the elastoplastic problem with the block-interfaces model as

$$\begin{aligned} &\min \quad \Pi^* = \frac{1}{2} d\boldsymbol{U}^T \boldsymbol{K} d\boldsymbol{U} - d\boldsymbol{U}^T (\Phi\lambda + \boldsymbol{q}) \\ &s.t. \quad \begin{cases} \boldsymbol{C} d\boldsymbol{U} - \boldsymbol{N}\lambda - \boldsymbol{d} + \nu = 0 \\ \nu^T \lambda = 0, \lambda \geq 0, \nu \geq 0 \end{cases} \end{aligned} \tag{28}$$

where $d\boldsymbol{U}$ is the global incremental displacement vector, λ is the plastic flow factor vector composed of the plastic flow factors of all blocks and interfaces and ν is the slack vector. $\boldsymbol{K}$ is the stiffness matrix same as that in equation (14), $\boldsymbol{q}$ the incremental load vector, $\boldsymbol{\Phi}$ the plastic potential matrix. Other matrices are

$$\mathbf{K} = \sum_{e=1}^{NE} tS_e \mathbf{D}_b + \sum_{j=1}^{NJ} t \int_{\beta_j} \mathbf{B}^T \mathbf{D}_s \mathbf{B} ds$$

$$\mathbf{q} = \sum_{m} \mathbf{Q}^T d\mathbf{P}_m + \sum_{e} \iint_{\Omega_e} \mathbf{Q}^T d\mathbf{f}_e dxdy$$

$$\Phi = \sum_{j=1}^{NJ_1} \sum_{a=1}^{L_j} t \int_{\beta_j} (\mathbf{V}_a^j \mathbf{B})^T ds + \sum_{e=1}^{NE_1} \sum_{a=1}^{L_e} t \iint_{\Omega_e} (\mathbf{Y}_a^e)^T dxdy$$

$$\mathbf{C} = \sum_{j=1}^{NJ_1} \sum_{a=1}^{L_j} \int_{\beta_j} \mathbf{Z}_a^j \mathbf{B} ds + \sum_{e=1}^{NE_1} \sum_{a=1}^{L_e} \iint_{\Omega^e} \mathbf{W}_a^e dxdy$$

$$\mathbf{N} = \sum_{j=1}^{NJ_1} \sum_{a=1}^{L_j} \int_{\beta_j} \mathbf{S}_a^j ds + \sum_{e=1}^{NE_1} \sum_{a=1}^{L_e} \iint_{\Omega_e} \mathbf{M}_a^e dxdy$$

$$\mathbf{d} = -\sum_{j=1}^{NJ_1} \sum_{a=1}^{L_j} \int_{\beta_j} \mathbf{f}_a^{s0j} ds - \sum_{e=1}^{NE_1} \sum_{a=1}^{L_e} \iint_{\Omega_e} \mathbf{f}_a^{b0e} dxdy$$

This is a quadratic programming with free variables and can be solved by the two steps algorithm [16].

CONTACT BEHAVIOR BETWEEN BLOCKS

The contact behavior between blocks can be solved similarly to the elastoplastic analysis. The contact displacement between rock blocks can be represented by the relative incremental displacements $d\delta_n$ and $d\delta_s$ between the top and bottom surfaces of interfaces in the normal and tangential direction. Each of the relative incremental displacements is composed of two parts, one is the relative incremental displacement $d\delta^e$ (elastic part) when rock blocks are in contact but with no relative slide, and the other is the relative sliding incremental displacement $d\delta^p$, namely

$$d\delta = d\delta^e + d\delta^p \tag{29}$$

The relation between the elastic relative incremental displacement $d\delta^e$ and the contact stress is similar to that of interfaces, that is

$$\begin{cases} d\sigma_n = K_n \cdot d\delta_n^e \\ d\tau_s = K_s \cdot d\delta_s^e \end{cases} \tag{30}$$

where K_n and K_s are the normal and tangential stiffness of the contact surfaces respectively; while $d\sigma_n$ and $d\tau_s$ are the normal and tangential contact stress respectively. The relation between the normal stress and the normal compressed displacement is nonlinear and it is simplified as two linear relations; namely, the normal stiffness of joints is assumed constant before joints are closed and assumed to be infinite after joints are closed completely. This relation can be simulated by the non-penetration condition (the normal compressed displacement should not be greater than the thickness of interfaces), namely

$$-\delta_n \le h \tag{31}$$

where δ_n is positive if the interface is in tensile state. The normal relative displacement δ_n

between rock blocks is composed of an elastic part and a plastic part. The elastic part is subject to Hook's law, so the non-penetration condition can be expressed as

$$-K_n^{-1}\boldsymbol{\sigma}_n - \boldsymbol{\delta}_n^p - h \le 0 \tag{32}$$

where δ_n is the relative sliding displacement between rock blocks.

The control equation of the contact problem between blocks can be stated as

$$\begin{aligned} f_1 &= \left|\tau_s\right| + \mu\sigma_n \le 0 && \textit{Coulomb law} \\ f_2 &= \sigma_n \le 0 && \textit{Contact condition} \\ f_3 &= -K_n^{-1}\sigma_n - \delta_n^p - h \le 0 && \textit{Nonpenetration condition} \end{aligned} \tag{33}$$

where μ is the friction coefficient. The above equation is nearly the same as the *Mohr-Coulomb* yield condition when the cohesion is equal to zero, so the contact problem between rock masses can be treated similarly with the *Mohr-Coulomb* yield condition. Rock blocks can only slide in the tangential direction of contact surfaces when the shear force reaches the sliding resistance force. Moreover, in order to satisfy the non-penetration condition, the normal elastic relative incremental displacement and the normal plastic incremental displacement should be equal but in opposite directions in each incremental step, so a suitable sliding potential function should be taken to get a normal plastic displacement which is in opposite direction to the normal elastic relative displacement. In this way, the yield function could not be chosen as the sliding potential function, namely, the contact problem is a non-associate plastic flow problem. The plastic potential functions should be chosen as

$$\begin{aligned} g_1 &= \left|\tau_s\right| + t \\ g_2 &= \sigma_n + t \\ g_3 &= K_n^{-1}\sigma_n + t \end{aligned} \tag{34}$$

in which t is a constant, g_1, g_2 and g_3 are the plastic potential functions for the contact

problems.

The relation between the relative sliding displacement $d\delta^p$ and the sliding potential function is

$$d\delta^p = \lambda \frac{\partial g}{\partial R} \tag{35}$$

In this way, we can integrate the contact behavior between rock masses with the elastoplastic problem of interfaces to simulate the nonlinear behavior of rock structures.

In the traditional incremental iteration method, contact conditions must be assessed in each step and many iterations are needed to satisfy all contact conditions. In this paper, the contact problem is treated similarly to the elastoplastic problem so that only a linear complementary problem needs to be solved by using the parametric variational principle. In this method, at most one Gaussian elimination is needed for each contact condition in each incremental step, so it provides better efficiency.

NUMERICAL EXAMPLES

The block-interfaces model is suitable to the nonlinear numerical simulation of both discontinuous and continuous structures. It can easily simulate the nonlinear behavior of discontinuous structures and will be reduced into an arbitrary convex polygon constant stress element for continuous structures. The following numerical examples are presented to illustrate the accuracy and effectiveness of the model.

(1) Linear elastic analysis of a jointed rock mass

Because of the presence of joints, the behavior of rock structures are largely controlled by the discontinuities. Singh [4] presented an equivalent anisotropic model for jointed rock masses which had given excellent results, except in the region of steep stress gradients near the loaded area.

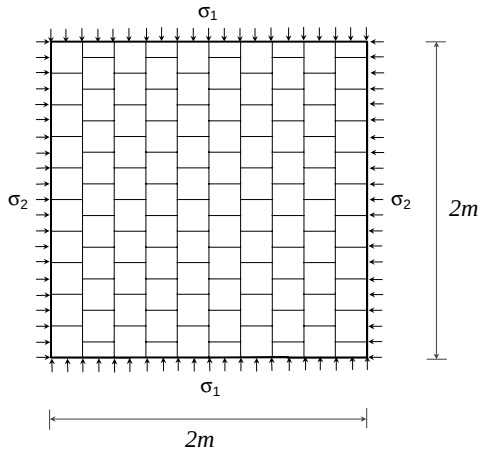

Figure 3 A jointed rock mass

Figure 3 shows a jointed rock mass which is intersected as a series of elastic rock blocks by two sets of staggered joints. The tangential and normal stiffnesses of the vertical joints are 438.8 *Gpa*/*m* and 395.9 *Gpa*/*m* respectively; while those of the horizontal joints 438.8 *Gpa*/*m* and 531.9 *Gpa*/*m*. The elastic modulus of rock blocks is 89 *Gpa*/*m* and the Poisson ratio ν_r = 0.26. According to theory given in Ref.[4] we can obtain the elastic moduli of the equivalent anisotropic continuum model as E_1=52.55 *Gpa*/*m*, E_2=41.9 *Gpa*/*m* and Poisson ratio ν_1=0.1535, ν_2=0.1224.

Assuming the jointed rock mass is in plane strain the relation of stress and strain is

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E_1}(1-\nu_1\nu_r)\sigma_1 - \frac{\nu_2}{E_2}(1+\nu_r)\sigma_2 \\ \varepsilon_2 &= -\frac{\nu_1}{E_1}(1+\nu_r)\sigma_1 + \frac{1}{E_2}(1-\nu_2\nu_r)\sigma_2 \end{aligned} \tag{36}$$

When we analyze the jointed rock mass in three load cases (see Table 1) using the block-interfaces model, the mean strains of the jointed rock masses are shown in Table 2. It can be seen that the results obtained by the block-interfaces model agree excellently with those obtained by the equivalent anisotropic continuum model. Compared with the equivalent

anisotropic continuum model, the discontinuities of jointed rock masses are explicitly represented in the block-interfaces model and it can be used to simulate the nonlinear mechanical behavior of rock structures.

Table 1 Load cases

1	Case 1	Case 2	Case 3
σ_1	10 *MPa*	0	10 *MPa*
σ_2	0	5 *MPa*	5 *MPa*

Table 2 The mean strain of the jointed rock mass ($\times 10^{-4}$)

	ε_1			ε_2		
	Case 1	Case 2	Case 3	Case1	Case2	Case 3
Block-interfaces model	-1.8333	0.18405	-1.6492	0.36809	-1.1553	-0.7872
Equivalent anisotropic model	-1.8269	0.184	-1.64288	0.368	-1.155	-0.787

(2) Elastoplastic and contact analysis of the jointed rock mass

Figure 4 shows the jointed rock mass subjected to the gravity load and a concentrated load *P* at point A. The material properties for joints and rock masses are the same as those in the first example. The specific gravity of the rock masses is 16 KN/m^3. The cohesion *C* of the joints is 20 KN/m^2 and the friction angle 0°. According to the plasticity theory the structure will be in a plastic limit state if the load *P* reaches 2*C*. The block-interfaces model is used to simulate the nonlinear behavior of the jointed rock mass and the relationship between the horizontal displacement at

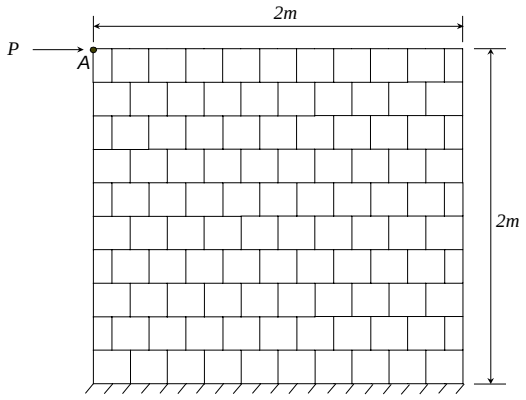

Figure 4 Rock Mass Subjected to a Concentrated Load

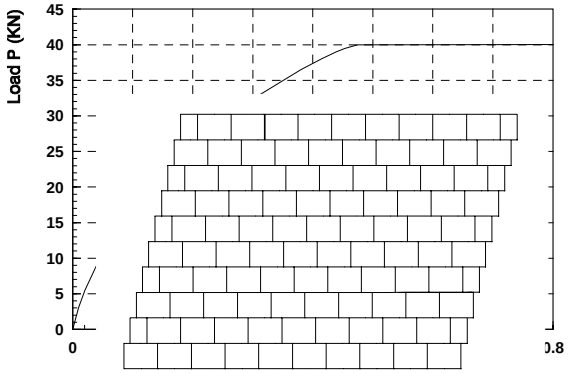

Figure 5 Displacement at the Top of the Rock Mass

Figure 6 Collapse Load of the Rock ($\times 10^{-5}m$)

point *A* and the load *P* is shown in Figure 5 which shows that the rock mass will collapse when the load *P* exceeds 40 KN and this coincides very well with the plasticity theory. The collapse mode of the rock mass is shown in Figure 6.

CONCLUSIONS

Based on the real characteristics of rock structures, the block-interfaces model is established by the constrained variational principle to simulate the mechanical behavior of discontinuum as well as continuum. The model can simulate the nonlinear behavior of discontinuous structure such as the failure of rock structures and foundations. The model has good precision and flexibility. The contact and elastoplastic behavior of rock structure are treated similarly and no equilibrium iterations are needed in each incremental step, so its computational effort is smaller.

In the block-interfaces model a large rock block should be divided into a number of smaller blocks connected by interfaces to get better results. The continuities between these smaller blocks can be enforced by the penalty function method or by introducing equation (8) into equation (28) as a constraint condition. A promising alternative for simulating a large block is the meshless method [19] in which no mesh except for a number of points is used within the block, so that the meshing of the blocks will be avoided. It will be discussed in future papers.

For three dimensional problems, the displacement at any point within a block can be obtained straightforwardly from equation (1), so the block-interfaces model can be easily extended into three dimensional problems.

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