Linear statics and free vibration sensitivity analysis of the composite sandwich plates based on a layerwise/solid-element method

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Abstract

Although many researches have been attracted to optimization problems of composite sandwich structures, there are rarely special literatures for sensitivity analysis which provides essential gradient information for the optimization. In this paper the linear statics and free vibration sensitivity analysis problems of the composite sandwich plates are studied based on a layerwise/solid-element method (LW/SE) which was developed in our previous work to eliminate or decrease the error induced by the equivalent methods of the core. In the present sensitivity analysis schemes the cores of the sandwich plates are discretized by three models, namely, full model, local model and equivalent model.

In the numerical examples, two kinds of sensitivity analysis schemes, the overall finite difference method (OFD) and the semi-analytical method (SAM), are employed to calculate the sensitivity coefficients of displacements, stresses and natural frequencies. The convergence is studied together with the effect of step size on the relative error. The performance of these three methods of modeling the honeycomb is investigated by using the sensitivity analysis scheme based on the local model and SAM.

1. Introduction

Sensitivity analysis of structures is a usual approach to obtain the gradient information of the response quantities with respect to the interest design parameters which include intrinsic variables like material properties and thickness, as well as geometric control variables governing the size and shape of the structures. In the past three decades, the sensitivity analysis has evolved as a major research area in structural analysis, holding out immense prospect for widespread applications, for instance, structural optimization, evaluation of structural reliability, and parameter identification. Composite sandwich structures are broadly used in many engineering because they offer a high bending stiffness with the minimum mass, the capability to be tailored, the high damping properties and the great potential for impact protection. Therefore, optimization design of this kind of structures is very important. Although many researchers have been attracted to this branch [1–4], there are rarely investigations for the sensitivity analysis which specially provides the essential gradient information for the optimization.

The past two decades have witnessed a spurt of research activities in the computational aspects of sensitivity analysis, such as the sensitivity analysis of static, eigenvalue, transient response and buckling problems. Generally, the sensitivity analysis consists of variational method and implicit differentiation method [5,6]. The variational methods, which is also referred to as the continuum methods, are based upon differentiating the continuum governing equations of the structural response. Although the continuum method generally used in the shape optimization of the continuous structures is mathematically rigorous, we can hardly use this method due to the difficulties in program coding and application. The implicit differentiation methods, which is also referred to as the discrete methods, are based upon the derivatives of the discrete formulations of the finite element methods or other numerical methods. With the rapid development of the finite element methods, the discrete methods are more and more popular than the continuum methods.

The existing discrete approaches such as analytical method (AM) [7], the overall finite difference methods (OFD) [8] and the semi-analytical method (SAM) [9–18] are commonly used. If the sensitivity analysis is implemented in finite element method (FEM), sensitivity calculations require the derivatives of the stiffness matrixes, the mass matrixes and the load vectors with...
respect to the design variables. In the AM methods, these derivatives are calculated analytically before the evaluation of the sensitivity coefficients. So the AM method provides useful physical insight into the effect of the variation of design or variation of some parameters on the structural response. But it is difficult to calculate the derivatives analytically in many cases, especially for the derivatives with respect to the geometric control variables [19]. The overall finite difference methods, in which the entire analysis is repeated for a perturbed variable, is popular since it is simple and accurate. However, the cost of calculation is very great for large structural systems. As to the semi-analytical approach, the differentiation of the component factors like the stiffness matrix, the load vector and so on is done approximately by finite difference methods, but the final solution procedure follows that of the analytical method. It can be implemented as easily as the OFD method and is as efficient as the SAM method. Thus the semi-analytical method is established based on the advantages of AM and OFD [20]. Obviously, both the OFD and the SA suffer truncation and condition errors which result from the finite difference methods, the magnitude of step size, and the machine accuracy [19].

Recently, the modeling scheme of composite sandwich structures is regarded as following the same analysis schemes of the composite laminated structures, such as the equivalent single layer theory (classical laminate theory and shear deformation laminated plate theories) [21–27], three-dimensional elastic theory (traditional 3-D elastic formulations, layerwise theory, unified formulation and generalized unified formulation) [28–32] and multiple model methods [33]. In the traditional analysis schemes of the composite sandwich structures [34–40], the core is firstly simplified as an equivalent anisotropic material and then modeled by the plates and shells theories. Their main disadvantage is that the equivalent core will result in large equivalent error especially in the key area and the thick core will further reduce the analysis accuracy of the plates and shells theories. For the composite stiffened laminated cylindrical shells, a layerwise/solid-element (LW/SE) method was established based on the layerwise theory and the finite element method (FEM)[41]. And then, for the composite sandwich plates this LW/SE method was extended to eliminate or decrease the error introduced by the equivalent methods about the core [42]. Furthermore, the detailed local deformation of the facesheets and core can be obtained by using this analysis scheme if the core cells belonging to the special attention area (for example, sheets and core can be obtained by using this analysis scheme if sandwich plates this LW/SE method was extended to eliminate or discretized laminated cylindrical shells, a layerwise/solid-element (LW/SE) method was established based on the layerwise theory and is as efficient as the AM method. Thus the semi-analytical method is established based on the advantages of AM and OFD [20]. Obviously, both the OFD and the SA suffer truncation and condition errors which result from the finite difference methods, the magnitude of step size, and the machine accuracy [19].

In the present work, the linear statics and free vibration sensitivity analysis problems of the composite sandwich plates are studied based on the LW/SE method. Two kinds of sensitivity analysis schemes SAM and OFD are employed to calculate the derivatives of the displacements, the stresses and the natural frequencies.

2. Mathematical formulations

2.1. A brief review of the LW/SE method

The schematic diagram of the LW/SE method for the composite sandwich structures is shown in Fig. 1, where the upper and lower facesheets are discretized with the four-noded quadrilateral elements and the layerwise theory, while the core is discretized with the eight-noded solid elements. Based on the compatibility conditions at the interface between facesheets and core, the layerwise theory can be conveniently coupled with the governing equations of the core established by the brick elements as a result of two characteristics of the layerwise theory. One is that the degree of freedoms (DOFs) of the layerwise theory is equal to that of the brick element, and another is that the displacements variables of the upper and lower surfaces of face sheets appear in the governing equations. Based on the finite element formulations of the face-sheets and core, the final governing equation of the composite sandwich structures can be assembled by using the compatibility conditions to ensure the continuity of displacements at the interface between facesheets and core. In the present work, the honeycomb is investigated. Three models, the full model, the local model and the equivalent model, are presented to model the honeycomb. In the full model all details of the honeycomb structures are discretized, as can be seen in Fig. 1b. In the equivalent model the honeycomb is firstly integrally considered as the anisotropic material by using some equivalent theories and then discretized by brick elements as shown in Fig. 1d. Although the equivalent model greatly reduces the computational cost and the difficulty of the algorithm, at the same time, compared to the full model it reduces the analysis accuracy and cannot obtain the detailed local deformation, such as that resulted from the point load and/or point supports. The local model illustrated in Fig. 1c, in which the honeycomb cells in the key region are modeled based on the real micro structure form completely instead of the equivalent anisotropic materials, is a combination of the full model and the equivalent model.

In the layerwise laminate theory[33], the displacements at point \((x,y,z)\) in the composite laminated plates are assumed to be

\[
u(x,y,z) = \sum_{i=1}^{N_z} \phi_i(x,y)\phi_i(z),\]

\[
u(x,y,z) = \sum_{i=1}^{N_z} p_i(x,y)\phi_i(z),\]

\[
u(x,y,z) = \sum_{i=1}^{N_z} W_i(x,y)\phi_i(z),\]

where \(u, v\) and \(w\) represent the displacement components in the \(x, y\) and \(z\) directions, respectively, \(\phi_i\) is a linear Lagrangian interpolation function through the thickness of the laminated facesheets of the composite sandwich plates. The laminate thickness dimension is subdivided into a series of \(N\) one-dimensional finite elements (\(Ne = N + 1\) nodes) whose nodes are located in planes parallel to \(xy\) plane in the undeformed laminated facesheets, \(u_i, v_i\) and \(w_i\) are the nodal values. \(N\) is also the number of mathematical layers of the laminated plates, which may be equal to or less than the number of physical layers.

As presented in the previous work, the final discrete equations of the composite sandwich structures are given by

\[
&M \hat{U} = K \hat{U} = F,
\]

where

\[
\begin{bmatrix}
M_{11} & M_{12} & M_{13} & 0 & M_{14} & M_{15} \\
M_{12} & M_{22} & 0 & 0 & 0 & M_{24} \\
0 & 0 & M_{33} & 0 & 0 & M_{35} \\
0 & 0 & 0 & M_{44} & 0 & M_{45} \\
0 & 0 & 0 & 0 & M_{55}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & 0 & K_{14} \\
K_{12} & K_{22} & 0 & 0 & K_{24} \\
0 & 0 & K_{33} & 0 & K_{35} \\
0 & 0 & 0 & K_{44} & 0 \\
0 & 0 & 0 & 0 & K_{55}
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5
\end{bmatrix}
\]

the superscript \(t\) denotes upper facesheet, and the superscript \(b\) denotes lower facesheet. For the facesheets, the subscripts 1 and
2 denote the interface displacements vector and the internal displacements vector, respectively. For the core, the subscripts 1, 2 and 3 denote the interface displacements vector related to the upper facesheet, the interface displacements vector related to the lower facesheet and the internal displacements vector, respectively.

\[
M_{a_{ij}}(i,j=1,2,3; a=t,b,c) \text{ is the mass matrices of the facesheets and the core,}
\]

\[
K_{a_{ij}}(i,j=1,2,3; a=t,b,c) \text{ is the stiffness matrixes of the facesheets and the core.}
\]

\[U, \dot{U} \text{ and } F \text{ are the displacements, accelerations and external loads vector, respectively.}
\]

For the static problem, Eq. (2) reduces to

\[KU = F. \quad (3)\]

For the free vibration problem, Eq. (2) reduces to

\[MU + Ku = 0. \quad (4)\]

The LW/SE method of the sandwich structures was established in the previous work [42]. And the modeling approach of the honeycomb was also investigated in detail for the static and free vibration analysis problem. The LW/SE method is reliable compared with the 3D elastic finite element method. The LW/SE method can obtain accurate displacements and stresses in the static problem of the composite laminated facesheets with various complex forms of core. For the static problem under concentrated load, the LW/SE method based on the full model possesses the best accuracy but is the most time-consuming, while the LW/SE method based on the equivalent model needs the least computational cost but provides the lowest accuracy, especially in the area nearby the concentrated load. Therefore, the LW/SE method based on local model has the balance between the accuracy and the computational burden. For free vibration analysis, however, the performances of the LW/SE methods based on the local model and the equivalent model are very close and differ slightly from the full model for the free vibration analysis. Different from the full model, the errors of the local model and the equivalent model are resulted from the equivalent material properties of the honeycomb. Since the natural frequencies represent the overall characteristic of the sandwich structures, it stands to reason that the performance of the LW/SE methods based on the local model and the equivalent model are very close for the free vibration analysis.

2.2. Sensitivity analysis based upon the semi-analytical method

Since the SAM method is attractive due to its general applicability, cost efficiency, and relative ease of implementation, it is chosen as a main sensitivity analysis scheme in this paper. The governing equations of the sensitivity coefficients for the static problem can be obtained by differentiating Eq. (3) with respect to \( \lambda \), where \( \lambda \) is the typical design parameter of the composite sandwich plates including intrinsic variables or geometric control variables. Differentiating of Eq. (3) with respect to \( \lambda \) leads to

\[
K \frac{\partial U}{\partial \lambda} = \frac{\partial F}{\partial \lambda} - \frac{\partial K}{\partial \lambda} K^{-1} F. \quad (5)
\]
If the load vector $\mathbf{F}$ is independent of the design parameter $\lambda$, namely $\partial \mathbf{F}/\partial \lambda = 0$, the sensitivity coefficients $\mathrm{Eq. (5)}$ for the static problem can be rewritten as

$$
\mathbf{K} \frac{\partial \mathbf{U}}{\partial \lambda} = - \frac{\partial \mathbf{K}}{\partial \lambda} \mathbf{K}^{-1} \mathbf{F}.
$$

(6)

For the facesheets of the composite sandwich plates, the strains associated with the hypothetical displacement fields \cite{33} in the small deformation problems can be calculated as follows

$$
e_{11} = \frac{\partial u}{\partial \lambda} = \sum_{i=1}^{N} \frac{w_i}{\partial \lambda} d_i \phi_i, \\
e_{22} = \frac{\partial v}{\partial \lambda} = \sum_{i=1}^{N} \frac{w_i}{\partial \lambda} d_i \phi_i, \\
e_{33} = \frac{\partial w}{\partial \lambda} = \sum_{i=1}^{N} w_i \frac{d_i}{\partial \lambda} \phi_i,
$$

(7)

$$
\gamma_{13} = \frac{\partial u}{\partial \lambda} + \frac{\partial w}{\partial \lambda} \sum_{i=1}^{N} \left( u_i \frac{d_i}{\partial \lambda} \phi_i + \frac{\partial w_i}{\partial \lambda} \phi_i \right), \\
\gamma_{12} = \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \lambda} \sum_{i=1}^{N} \left( u_i \frac{d_i}{\partial \lambda} \phi_i + \frac{\partial v_i}{\partial \lambda} \phi_i \right), \\
\gamma_{23} = \frac{\partial v}{\partial \lambda} + \frac{\partial w}{\partial \lambda} \sum_{i=1}^{N} \left( v_i \frac{d_i}{\partial \lambda} \phi_i + \frac{\partial w_i}{\partial \lambda} \phi_i \right),
$$

where $e_{11}, e_{22}, e_{33}, \gamma_{12}, \gamma_{13},$ and $\gamma_{23}$ are the strain components.

According to $\mathrm{Eq. (7)}$, if the design parameter $\lambda$ has no relationship with $\phi_i$, the derivatives of strains are given by

$$
\frac{\partial e_{11}}{\partial \lambda} = \sum_{i=1}^{N} \frac{\partial}{\partial \lambda} \frac{\partial u_i}{\partial \lambda} \phi_i, \\
\frac{\partial e_{22}}{\partial \lambda} = \sum_{i=1}^{N} \frac{\partial}{\partial \lambda} \frac{\partial v_i}{\partial \lambda} \phi_i, \\
\frac{\partial e_{33}}{\partial \lambda} = \sum_{i=1}^{N} \frac{\partial}{\partial \lambda} \frac{\partial w_i}{\partial \lambda} \phi_i, \\
\frac{\partial \gamma_{13}}{\partial \lambda} = \sum_{i=1}^{N} \left( \frac{\partial u_i}{\partial \lambda} \phi_i + \frac{\partial w_i}{\partial \lambda} \phi_i \right), \\
\frac{\partial \gamma_{12}}{\partial \lambda} = \sum_{i=1}^{N} \left( \frac{\partial u_i}{\partial \lambda} \phi_i + \frac{\partial v_i}{\partial \lambda} \phi_i \right), \\
\frac{\partial \gamma_{23}}{\partial \lambda} = \sum_{i=1}^{N} \left( \frac{\partial v_i}{\partial \lambda} \phi_i + \frac{\partial w_i}{\partial \lambda} \phi_i \right).
$$

(8)

Then, the derivative of stress with respect to the design parameter $\lambda$ is given by

$$
\frac{\partial \sigma}{\partial \lambda} = \frac{\partial \mathbf{D}}{\partial \lambda} \varepsilon + \mathbf{D} \frac{\partial \varepsilon}{\partial \lambda},
$$

(9)

where $\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{13}, \tau_{23}]^T$, $\varepsilon = [e_{11}, e_{22}, e_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23}]^T$, $\mathbf{D} = [C_{ij}]$, and $D_{ij}(i, j) = 1, 2, \ldots, 6$ denote the elasticity coefficients of material.

If the design parameter $\lambda$ is independent of the matrix $\mathbf{D}$, namely $\partial \mathbf{D}/\partial \lambda = 0$, $\mathrm{Eq. (9)}$ can be rewritten as

$$
\frac{\partial \sigma}{\partial \lambda} = \mathbf{D} \frac{\partial \varepsilon}{\partial \lambda}.
$$

(10)

For the core of the sandwich structures, the sensitivity coefficients of the strains and the stresses can be obtained by the same method of the facesheets.

As shown in $\mathrm{Eq. (4)}$, the free vibration problem is governed by

$$
(K - \omega_n^2 \mathbf{M}) \mathbf{q}_n = 0,
$$

(11)

where $\omega_n$ and $\mathbf{q}_n$ are the natural angular frequency and the associated vibration mode shape vector for mode $n$, respectively.

Differentiating the free vibration governing equation with respect to the design variable $\lambda$ leads to the frequency sensitivity expression

$$
\frac{\partial \omega_n}{\partial \lambda} = \frac{1}{2 \omega_n^2} \mathbf{q}_n^T \left( \frac{\partial \mathbf{K}}{\partial \lambda} - \omega_n^2 \frac{\partial \mathbf{M}}{\partial \lambda} \right) \mathbf{q}_n.
$$

(12)

If the vibration mode shape $\phi_n$ is normalized, so that $\phi_n^T M \phi_n = 1$, the sensitivity coefficients of the natural frequencies can be further simplified into

$$
\frac{\partial \omega_n}{\partial \lambda} = \frac{1}{2 \omega_n^2} \mathbf{q}_n^T \left( \frac{\partial \mathbf{K}}{\partial \lambda} - \omega_n^2 \frac{\partial \mathbf{M}}{\partial \lambda} \right) \mathbf{q}_n.
$$

(13)

The SAM method was used in shape optimization by Ziel-nkiewicz and Campbell in 1973 \cite{9}. Cheng and Liu \cite{11} named their SAM formulation as the Quasi-Analytical Method, and demonstrated the advantages in terms of efficiency, accuracy and ease of implementation. However, as certain accuracy problems were soon reported \cite{19,43}, the most important challenge for the practical usage is to reduce the error. It was observed that, for size design variables and the material properties, the SAM performed accurately. There are two more sources of error for this sensitivity analysis technique. One source is the condition and truncation errors of the finite differencing. For some shape variables, the method proves to be very sensitive to the step size, and in some cases, no step size gives accurate derivatives. Another source of error for this method is poor finite element modeling. It is obvious that, if the finite element model is not good enough for producing accurate values of the structural response, the calculations of their sensitivities will also be erroneous.

In order to improve the accuracy, Cheng et al. \cite{44} studied the accuracy problems of the structures composed of the beam elements, the plane stress triangular elements, the plane stress iso-parametric elements, the solid elements and the triangular bending elements. Based on those investigations, an alternative forward/backward difference scheme was proposed instead of the usual forward difference scheme. The idea of this method is that a severe error appears only when the number of elements is sufficiently large, and in such case, by making use of the alternative finite difference scheme, the error introduced in one element cancels out the error introduced in its neighboring element to a great extent. Olhoff and Rasmussen \cite{45} studied the inaccuracy problem through an exact analysis of a modeling problem that might be considered as a simplified type of shape optimization problem. And a method was established based on a non-standard finite difference approximation for computation of the derivatives of the stiffness matrix. The approximate derivative was then expressed as the mean value of the first order forward and backward difference computations. So the dependence of the errors on the number of the finite elements was completely removed. Then, Olhoff et al. \cite{13} proposed an “exact” numerical differentiation scheme using some correction factors and certain special element functions, where the correction factors could be evaluated as an initial step of the sensitivity analysis and the special element functions were defined by incomplete polynomials which depended on the design variables linearly. Thus, this finite difference scheme yielded exact derivatives.

2.3. Sensitivity analysis based upon the overall finite difference methods

Similarly, both of the forward-difference method and the central-difference method can also be employed in the sensitivity analysis schemes based upon the OFD.
The forward-difference method for the displacements and natural frequencies can be shown as follows

\[
\frac{\partial \mathbf{U}}{\partial \lambda} = \frac{\mathbf{U}(\lambda + \Delta \lambda) + \mathbf{U}(\lambda)}{\Delta \lambda},
\]

\[
\frac{\partial \omega_n}{\partial \lambda} = \frac{\omega_n(\lambda + \Delta \lambda) + \omega_n(\lambda)}{\Delta \lambda}.
\]

(14)

And the central-difference method for the displacements and natural frequencies can be shown as follows

\[
\frac{\partial \mathbf{U}}{\partial \lambda} = \frac{\mathbf{U}(\lambda + \Delta \lambda/2) + \mathbf{U}(\lambda - \Delta \lambda/2)}{\Delta \lambda},
\]

\[
\frac{\partial \omega_n}{\partial \lambda} = \frac{\omega_n(\lambda + \Delta \lambda/2) + \omega_n(\lambda - \Delta \lambda/2)}{\Delta \lambda}.
\]

(15)

The forward-difference method is adopted in this paper. For many applications, this simple method of finite difference approximations provides sufficient accuracy for an adequate design derivative evaluation, presuming a proper value of the perturbation \([15]\). Other approaches that have been suggested in the literatures, such as the central difference method, and Richardson extrapolation can be applied as alternative strategies, when the forward difference method fails to deliver results with acceptable accuracy owing to large condition errors. The later methods, however, imply the drawback that the system must be calculated several times, depending upon the desired degree of accuracy.

2.4. Evaluation method of the derivatives of the stiffness and mass matrices

Based on the SAM method, the nodal displacements sensitivity coefficients \(\partial \mathbf{U} / \partial \lambda\), and the natural frequencies sensitivity coefficients \(\partial \omega_n / \partial \lambda\) can be calculated with Eqs. (6) and (13), respectively. However, we have to firstly obtain the derivatives of the stiffness matrix \(\partial \mathbf{K} / \partial \lambda\) and the mass matrix \(\partial \mathbf{M} / \partial \lambda\). Note that because the design parameter \(\lambda\) belonging to the facesheets does not affect the stiffness matrix and the mass matrix of the honeycomb, the calculation of \(\partial \mathbf{K} / \partial \lambda\) and \(\partial \mathbf{M} / \partial \lambda\) can be skipped. Similarly, because the design parameter \(\lambda\) belonging to the honeycomb does not affect the stiffness matrix and the mass matrix of the facesheets, the calculation of \(\partial \mathbf{K} / \partial \lambda\) and \(\partial \mathbf{M} / \partial \lambda\) can be skipped. So the derivatives of the stiffness matrix and the mass matrix can be calculated as follows.

For the design parameter \(\lambda\) that belongs to the facesheets

\[
\frac{\partial \mathbf{K}}{\partial \lambda} = \begin{bmatrix}
K_{11,1}^i & K_{12,1}^i & 0 & 0 & 0 \\
K_{21,1}^i & K_{22,1}^i & 0 & 0 & 0 \\
0 & 0 & K_{11,2}^c & K_{12,2}^c & 0 \\
0 & 0 & K_{21,2}^c & K_{22,2}^c & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\frac{\partial \mathbf{M}}{\partial \lambda} = \begin{bmatrix}
M_{11,1}^i & M_{12,1}^i & 0 & 0 & 0 \\
M_{21,1}^i & M_{22,1}^i & 0 & 0 & 0 \\
0 & 0 & M_{11,2}^c & M_{12,2}^c & 0 \\
0 & 0 & M_{21,2}^c & M_{22,2}^c & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(16)

For the design parameter \(\lambda\) that belongs to the honeycomb

\[
\frac{\partial \mathbf{K}}{\partial \lambda} = \begin{bmatrix}
K_{11,1}^i & 0 & K_{12,1}^c & 0 & K_{13,1}^c \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{12,2}^c & 0 & K_{13,2}^c \\
K_{21,1}^i & 0 & K_{22,1}^c & 0 & K_{23,1}^c \\
0 & 0 & 0 & 0 & 0 \\
K_{31,1}^i & 0 & K_{32,1}^c & 0 & K_{33,1}^c \\
\end{bmatrix},
\]

\[
\frac{\partial \mathbf{M}}{\partial \lambda} = \begin{bmatrix}
M_{11,1}^i & 0 & M_{12,1}^c & 0 & M_{13,1}^c \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{21,2}^c & 0 & M_{22,2}^c \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{31,2}^c & 0 & M_{32,2}^c \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(17)

According to the approaches of evaluating the derivatives of the stiffness matrix and the mass matrix, the sensitivity analysis schemes are divided into two classes. If those derivatives are determined analytically, the sensitivity analysis scheme is called analytical method (AM). If those derivatives are determined by finite differentiation, the term semi-analytical method (SAM) is applied. While the AM is expedient and has been applied successfully in the problems involving intrinsic variables, it is often a formidable task.

Fig. 2. In-plane finite element discretization of the composite sandwich plates. (a) Full model discretization of the facesheets and honeycomb; (b) local model discretization of the facesheets and honeycomb; and (c) equivalent model discretization of the facesheets and honeycomb.
to implement this method when the geometric control variables are encountered. For the problems studied in this paper, the element stiffness of the facesheets developed by layerey theory is very complex especially if the nonlinear strain–stress relations are considered, so it is much more attractive to use the SAM. Here again, both of the forward-difference method and the central-difference method can also be employed to calculate $K_{ij}$ and $M_{ij}$.

In the forward-difference method,

$$K_{ij} = \frac{K_{ij}(\Delta \lambda) - K_{ij}^{(0)}}{\Delta \lambda} \quad (i,j = 1,2; x = t, b),$$

$$K_{ij}^{(0)} = \frac{K_{ij}^{(0)}(\Delta \lambda) - K_{ij}^{(0)}}{\Delta \lambda} \quad (i,j = 1,2, 3),$$

$$M_{ij} = \frac{M_{ij}(\Delta \lambda) - M_{ij}^{(0)}}{\Delta \lambda} \quad (i,j = 1,2, \alpha = t, b),$$

$$M_{ij}^{(0)} = \frac{M_{ij}(\Delta \lambda) - M_{ij}^{(0)}}{\Delta \lambda} \quad (i,j = 1,2, 3).$$

In the central-difference method,

$$K_{ij} = K_{ij}^{(0)}(\lambda + \Delta \lambda/2) - K_{ij}^{(0)}(\lambda - \Delta \lambda/2) \quad \frac{\Delta \lambda}{(i,j = 1,2, \alpha = t, b),}$$

$$K_{ij}^{(0)} = K_{ij}^{(0)}(\lambda + \Delta \lambda/2) - K_{ij}^{(0)}(\lambda - \Delta \lambda/2) \quad \frac{\Delta \lambda}{(i,j = 1,2, 3),}$$

$$M_{ij} = M_{ij}^{(0)}(\lambda + \Delta \lambda/2) - M_{ij}^{(0)}(\lambda - \Delta \lambda/2) \quad \frac{\Delta \lambda}{(i,j = 1,2, \alpha = t, b),}$$

$$M_{ij}^{(0)} = M_{ij}^{(0)}(\lambda + \Delta \lambda/2) - M_{ij}^{(0)}(\lambda - \Delta \lambda/2) \quad \frac{\Delta \lambda}{(i,j = 1,2, 3).}$$

3. Numerical examples

3.1. Accuracy of the present sensitivity analysis schemes

The purpose of this section is to investigate the effects of the step size of the SAM and the OFD on the accuracy of the sensitivity coefficients, and find an optimum step size for the sensitivity analysis in the next sections, in case that large step sizes would generate large truncation error and small step sizes would amplify the round-off errors. In general, for the plate or shell structures a step size in the range of $10^{-4} \lambda - 10^{-2} \lambda$ is found to be able to yield satisfactory results for the OFD while for the SAM a step size value below $10^{-2} \lambda$ is sometimes required to guarantee accuracy [16,46].

3.1.1. Step sizes of the SAM and the FDM

For the composite sandwich plates employed in this numerical example, the in-plane finite element discretization of the facesheets and honeycomb is shown in Fig. 2, where the discretization of the upper and lower facesheets should be consistent with that of the honeycomb in the interface. The composite sandwich plate with four clamped edges is subjected to a central point load of the magnitude $F = 1$ N on the upper facesheet. The sizes of regular hexagons honeycomb cells are $l = h = 4$ mm, $H = 6$ mm, and $t_c = 0.3464$ mm, as shown in Fig. 3. Material properties of the honeycomb are taken as $E_t = 68,000$ MPa, $\nu = 0.3$ and $\rho_t = 2700$ kg/m$^3$. The equivalent material properties of the honeycomb are taken as $E_{11} = E_{22} = 101.7558$ MPa, $E_{33} = 5888.8$ MPa, $G_{23} = G_{13} = 1307.7$ MPa, $G_{12} = 25.4978$ MPa.

Table 1

<table>
<thead>
<tr>
<th>$t/10^{-4}$</th>
<th>$E_t/10^{-10}$</th>
<th>$\Delta x$</th>
<th>Full model</th>
<th>Local mode</th>
<th>Equivalent model</th>
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<tr>
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<td>FDM</td>
<td>SAM</td>
<td>FDM</td>
<td>SAM</td>
</tr>
<tr>
<td>$10^{-1}$</td>
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<td>-1.82415</td>
<td>-1.77361</td>
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<td>(-9.0758)</td>
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<td>(-9.0760)</td>
<td>(-5.8227)</td>
<td>(-9.1083)</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>-1.91787</td>
<td>-1.92499</td>
<td>-1.93137</td>
<td>-1.93856</td>
<td>-0.51632</td>
</tr>
<tr>
<td></td>
<td>(-0.9881)</td>
<td>(-0.6205)</td>
<td>(-0.9884)</td>
<td>(-0.6198)</td>
<td>(-0.9933)</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>-1.93508</td>
<td>-1.93580</td>
<td>-1.94871</td>
<td>-1.94943</td>
<td>-0.52097</td>
</tr>
<tr>
<td></td>
<td>(-0.9996)</td>
<td>(-0.6625)</td>
<td>(-0.9995)</td>
<td>(-0.6623)</td>
<td>(-0.1016)</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>-1.93682</td>
<td>-1.93689</td>
<td>-1.95046</td>
<td>-1.95053</td>
<td>-0.52144</td>
</tr>
<tr>
<td></td>
<td>(-0.0098)</td>
<td>(-0.0062)</td>
<td>(-0.0097)</td>
<td>(-0.0062)</td>
<td>(-0.0115)</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>-1.93699</td>
<td>-1.93700</td>
<td>-1.95063</td>
<td>-1.95064</td>
<td>-0.52149</td>
</tr>
<tr>
<td></td>
<td>(-0.0100)</td>
<td>(-0.0005)</td>
<td>(-0.0100)</td>
<td>(-0.0005)</td>
<td>(-0.0019)</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>-1.93701</td>
<td>-1.93701</td>
<td>-1.95065</td>
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<td>-0.52150</td>
</tr>
<tr>
<td></td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>-1.93701</td>
<td>-1.93701</td>
<td>-1.95065</td>
<td>-1.95065</td>
<td>-0.52150</td>
</tr>
<tr>
<td></td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
<td>(-0.0000)</td>
</tr>
</tbody>
</table>

Fig. 3. Cell of a honeycomb core.


\(v_{23} = 0.0, \quad v_{12} = 1.0, \quad v_{31} = 0.3, \quad \rho_0^B = 360 \text{ kg/m}^3,\) and the equivalent method can be found in the Ref. [47]. The material properties of the facesheets are taken as \(E_{11} = 156,500 \text{ MPa}, \quad E_{22} = E_{33} = 13,000 \text{ MPa}, \quad G_{12} = G_{13} = 4540 \text{ MPa}, \quad G_{23} = 6960 \text{ MPa}, \quad v_{23} = 0.4, \quad v_{12} = v_{13} = 0.23, \quad \rho_s = 2700 \text{ kg/m}^3.\) The stacking sequence of the upper and lower facesheets is [0/90/0], and the thickness of the facesheets \(t_i = 1 \text{ mm}.

Convergence of the maximum sensitivity coefficients of the displacement \(w\) in lower surface of the upper facesheet and the fundamental frequency with respect to the step size of the SAM and FDM are listed in Tables 1, 2 respectively, where the values in the brackets are the relative errors given by Eq. (20). \(\Delta E^\text{eq} = \Delta \lambda / \lambda\) is the equivalent step size, and \(\Delta \lambda\) is the real step size in the finite difference scheme. It can be seen from Tables 1, 2 that with respect to the parameters \(E_{11}\) and \(t_i\) both of the SAM and the OFD converge to a stable value which is regarded as the true solution of the sensitivity coefficients. The convergence speed of the OFD is faster than that of the SAM, which means that the step size required in the sensitivity analysis based on the OFD is smaller than that required in the sensitivity analysis based on the SAM. The convergence speed of the static response sensitivity analysis (displacements) is faster than that of the free vibration sensitivity analysis (natural frequencies). The sensitivity analysis schemes based on the full model, the local model and the equivalent model have same convergence with respect to the step size of the SAM and FDM. Consequently, for the composite sandwich plates studied in this paper a step size in the range of \(10^{-2} - 10^{-5}\) is required for the OFD while for the SAM a step size value \(10^{-5}\) is required to guarantee the relative error below 0.01%. Although the OFD can be simply coded and has high convergence speed, the computational cost of the SAM is almost half of the OFD and the SAM can also obtain high precision results with the smaller step size. So the SAM with a step size value \(10^{-3}\) is chosen in the following numerical examples.

Furthermore, as shown in Tables 1 and 2, when the step sizes of the SAM and the OFD reduce one order of magnitude, the relative

Table 2
Convergence of the sensitivity coefficients of the fundamental frequency with respect to the step size of the SAM and FDM.

<table>
<thead>
<tr>
<th>(\Delta \lambda)</th>
<th>(\Delta E^\text{eq})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>(10^{-1})</td>
</tr>
<tr>
<td>(10^{-2})</td>
<td>(10^{-2})</td>
</tr>
<tr>
<td>(10^{-3})</td>
<td>(10^{-3})</td>
</tr>
<tr>
<td>(10^{-4})</td>
<td>(10^{-4})</td>
</tr>
<tr>
<td>(10^{-5})</td>
<td>(10^{-5})</td>
</tr>
<tr>
<td>(10^{-6})</td>
<td>(10^{-6})</td>
</tr>
<tr>
<td>(10^{-7})</td>
<td>(10^{-7})</td>
</tr>
</tbody>
</table>

\(\alpha = 10^{-6} \times \text{kg/m}^3, \quad k = 10^{-3} \times \text{N/m}, \quad t = 10^{-1}\)

Fig. 4. The curves of the logarithmic relative error and the equivalent logarithmic step size, (a) Displacements and (b) fundamental frequency.
errors of the displacements and the fundamental frequency sensitivity coefficients also reduce about one order of magnitude. The relative errors in the 1 and 2 are defined as following

\[ e_{\text{relative}} = e_{\text{truncation}} + e_{\text{roundoff}} = \left( \frac{\Delta f}{f} + \frac{\partial f}{\partial x} \right) \times 100. \]  

where \( f \) denotes the response quantities such as the displacements and the natural frequencies. Here, \( \partial f/\partial x \) is the sensitivity coefficients which are obtained by the SAM and the OFD with the step size \( 10^{-7} \).

According to the Taylor’s series, the increment of the response quantities results from

\[ \Delta f \approx \frac{\partial f}{\partial x} \Delta x. \]  

Fig. 5. Central line distributions of the displacement sensitivity coefficients of the upper facesheet obtained by the full model, the local model and the equivalent model with respect to the design variable \( E_{11} \). (a) Upper surface, (b) interface between the second layer and third layer, (c) interface between the first layer and second layer, and (d) lower surface
\[ \Delta f(\lambda) = \frac{\partial f}{\partial \lambda} \Delta \lambda + O(\Delta \lambda^2). \] (21)

Substituting Eq. (21) into (20), leads to the relationship between the relative truncation error and the step size

\[ \varepsilon_{\text{truncation}} = \left( \frac{\partial f}{\partial \lambda} \Delta \lambda + O(\Delta \lambda^2) \right) \frac{\partial f}{\partial \lambda} = O(\Delta \lambda) / \partial f \] (22)

Taking the logarithm of the both sides of Eq. (22)

\[ \log\varepsilon_{\text{truncation}} = \log(\Delta \lambda) - \log(\partial f / \partial \lambda) - \log(\Delta \lambda_{\text{eq}}) + \log(\lambda) - \log(\partial f / \partial \lambda). \] (23)

It can be seen from Eq. (23) that the logarithmic relative truncation error \( \log\varepsilon_{\text{truncation}} \) has linear relation with the equivalent logarithmic step size \( \log(\Delta \lambda_{\text{eq}}) \), and the slope of those curves is 1. The \( \log\varepsilon_{\text{relative}} - \log(\Delta \lambda_{\text{eq}}) \) curves for the displacements and the fundamental frequency with respect to the parameter \( E_{11} \) are shown in Fig. 4. When the step size \( \Delta \lambda > 10^{-4} \), the logarithmic \( \log\varepsilon_{\text{relative}} \) has good linear relation at the equivalent logarithmic step size \( \log(\Delta \lambda_{\text{eq}}) \), and the slope of those curves is also about 1. Since the truncation error is more remarkable than around-off error for the large step size, the convergence of the error in the Tables 1,2 is in good agreement with the theoretical convergence given in Eq. (23). The increase of the round-off error results in the decrease of the slope when the step size \( \Delta \lambda > 10^{-5} \).

3.1.2. Performance of the full model, the local model and the equivalent model for sensitivity analysis

It is obvious from Tables 1,2 that with respect to the parameters of facesheets \( E_{11} \) and \( t_s \), the results obtained by the full model and the local model are very close and differ obviously from the results of the equivalent model for the displacements sensitivity coefficients, while for the natural frequencies sensitivity coefficients the local model and the equivalent model are very close and differ slightly from the results of the full model. It is very similar to the conclusion about the static response and the free vibration analysis in the previous work. A detailed comparative study will be conducted in this section to investigate the performance of those three modeling schemes for static sensitivity analysis and the free vibration sensitivity analysis problems. The composite sandwich plate in Section 3.1.1 is also employed.

The displacements sensitivity coefficients of the upper facesheet along the central line obtained by three models are shown in Fig. 5 with respect to the parameters of facesheets \( E_{11} \), and the in-plane distributions of the displacements sensitivity coefficients

**Fig. 6.** In-plane distribution of the displacement sensitivity coefficients of the upper facesheet obtained by the full model, local model and equivalent model with respect to the design variable \( E_{11} \). (a) Full model; (b) local model; and (c) equivalent model.
of the upper facesheets are shown in Fig. 6. It can be observed from Figs. 5 and 6 that compared with the full model the local model can provide more accurate displacements sensitivity coefficients in entire problem domains of the upper facesheet, especially in the local area around the concentrated load and the top/bottom surfaces. The displacements sensitivity coefficients obtained by the equivalent model are in good agreement with those obtained by the full model in the area away from the concentrated load, while in the local area around the concentrated load the sensitivity scheme based on the equivalent model cannot provide correct results in the upper facesheet.

The displacements sensitivity coefficients of the lower facesheet along the central line obtained by three models are shown in Fig. 7 with respect to the parameters of facesheets $E_{11}$, and the in-
Table 3
Maximum displacement sensitivity coefficients of the honeycomb obtained by the full model, the local model and the equivalent model with respect to the parameter $E_{11}$.

<table>
<thead>
<tr>
<th>$z$ (mm)</th>
<th>Full model</th>
<th>Local model</th>
<th>Equivalent model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u/10^{-11}$</td>
<td>$v/10^{-12}$</td>
<td>$w/10^{-12}$</td>
</tr>
<tr>
<td>1</td>
<td>0.75692</td>
<td>8.69463</td>
<td>-4.24945</td>
</tr>
<tr>
<td>2</td>
<td>0.32172</td>
<td>3.37981</td>
<td>-4.47298</td>
</tr>
<tr>
<td>3</td>
<td>0.43094</td>
<td>2.14438</td>
<td>-4.59915</td>
</tr>
<tr>
<td>4</td>
<td>1.51222</td>
<td>7.77873</td>
<td>-5.09848</td>
</tr>
</tbody>
</table>

Table 4
Sensitivity coefficients of the first 8 natural frequencies obtained by the full model, the local model and the equivalent model with respect to the parameter $E_{11}$.

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Full model/10^{-4}</th>
<th>Local model/10^{-4}</th>
<th>Equivalent model/10^{-4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.36663</td>
<td>4.14847</td>
<td>4.61409</td>
</tr>
<tr>
<td>2</td>
<td>2.77901</td>
<td>3.28245</td>
<td>3.64119</td>
</tr>
<tr>
<td>3</td>
<td>2.71671</td>
<td>3.27316</td>
<td>3.45857</td>
</tr>
</tbody>
</table>

Fig. 8. In-plane distribution of the displacement sensitivity coefficients of the lower facesheet obtained by the full model, local model and equivalent model with respect to the design variable $E_{11}$. (a) Full model; (b) local model; and (c) equivalent model.
Fig. 9. The first 8 modal shape obtained by the full model, the local model and the equivalent model. (a) Full model; (b) local model and (c) equivalent model.
plane distributions of the displacement sensitivity coefficients of the lower facesheets are shown in Fig. 8. It can be obvious from Figs. 7 and 8 that for the displacements $u$ and $v$ the sensitivity results obtained by the local model and the equivalent model are in good agreement with those obtained by the full model in entire problem domains of the lower facesheet especially in the local area around the concentrated load. For displacement $w$ the sensitivity results obtained by the local model are in good agreement with those obtained by the full model in entire problem domains. Although the sensitivity results of the displacement $w$ obtained by the equivalent model are in good agreement with those obtained by the full model in the area away from the concentrated load, it cannot provide correct results in the local area around the concentrated load for the displacement $w$ in the lower facesheet. Therefore, in the local area around the concentrated load where the honeycomb cells are modeled based on the real structure form completely, the sensitivity analysis scheme based on the equivalent model in the lower facesheet is more accurate than in the upper facesheet. One important reason is that local effect resulted from the concentrated load is not significant in the lower facesheets because the honeycomb transfers dispersedly the concentrated load from the upper facesheet to the lower facesheet.

In the $xy$ plane of the upper facesheet the displacements $u$ and $v$ in the area near the central point are most sensitive to $E_{11}$, while the maximum sensitivity coefficient of the displacement $w$ appears at the central point. In the $xy$ plane of the lower facesheet the displacement $u$ in the area near the edges ($x \approx 0$, $42$ mm and $y \approx 0$ mm) is most sensitive to $E_{11}$, the displacement $v$ in the area nearby the points ($x \approx 0$ mm and $y \approx 15$, $30$ mm) is most sensitive, and the maximum sensitivity coefficient of the displacement $w$ appears in the area nearby the central point. The concentration effect of the displacements sensitivity coefficients in the upper facesheet is more significant than that in the lower facesheet. The upper facesheet is subjected to a concentrated load in this numerical example, but the lower facesheet is subjected to a distributed load due to the existence of the honeycomb. The most important reason is that the distributions of the displacement sensitivity coefficients in the upper and lower facesheets are different slightly. In addition, the sensitivity analysis schemes based on the full model and the local model can obtain the local concentration effect of the displacements sensitivity coefficients especially in the upper facesheet, but the local model obviously reduces the computational cost.

With respect to the parameter $E_{11}$, in the thickness direction the maximum displacement sensitivity coefficients of the honeycomb obtained by three models are listed in Table 3, where $z$ is the coordinate in the thickness direction. The displacement sensitivity coefficients of the honeycomb obtained by the local model are in good agreement with those obtained by the full model. Except the displacement $w$, the error of the equivalent model is not acceptable.

The sensitivity coefficients of the first 8 natural frequencies obtained by three models with respect to the parameter $E_{11}$ are listed in Table 4. And the first 8 modal shapes obtained by the full model, the local model and the equivalent model are shown in Fig. 9. The sensitivity coefficients of the first 4 natural frequencies calculated by the local model and the equivalent model are very close and differ slightly from the results of the full model. Compared to the full model, the errors of the local model and the equivalent model mainly result from the equivalent material properties of the honeycomb. Since the natural frequencies represent the overall characteristics of the sandwich structures, it stands to reason that the performances of the local model and the equivalent model are very close for the free vibration sensitivity analysis, especially the lower natural frequencies. However, three models have obviously different structural characteristics, so the sensitivity analysis results of the higher natural frequencies obtained by the local model and the equivalent model are inaccurate compared with the results obtained by the full model. Even so the sensitivity analysis scheme based on the local model still can be used to investigate the influence of the design parameters on the lower natural frequencies, and then to find the primary factors which have significant effect on the free vibration characteristics of the composite sandwich plates.

![Fig. 10. The composite sandwich plates with the very soft honeycomb cores at the central.](image)

<p>| Table 5 |</p>
<table>
<thead>
<tr>
<th>Sensitivity coefficients of the first 8 natural frequencies obtained by the full model, the local model and the equivalent model with respect to the parameter $E_{11}$. The honeycomb cores at the central are softer than those at other area of the composite sandwich structure.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mode number</strong></td>
</tr>
<tr>
<td>Local model $10^{-4}$</td>
</tr>
<tr>
<td>Equivalent model $10^{-4}$</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>
Fig. 11. The first 8 modal shape obtained by the full model, the local model and the equivalent model. The honeycomb cores at the central are softer than those at other area of the composite sandwich structure. (a) Full model; (b) local model and (c) equivalent model.
For the composite sandwich plates with the very soft honeycomb cores at the central (the region of the very soft cores are shown in Fig. 10), the sensitivity coefficients of the first 8 natural frequencies obtained by three models with respect to the parameter $E_{11}$ are listed in Table 5. The Young's modulus of the very soft cores is 6800 MPa, and the Possion's rate is 0.3. The first 8 modal shapes obtained by the full model, the local model and the equivalent model are shown in Fig. 11. The numbers of the last line in Table 5 are the real mode numbers of the frequencies obtained by using equivalent model. It can keep the modal shape consistent in each model. It can be seen from Table 5 that the sensitivity coefficients of the first 6 natural frequencies calculated by the local model and the equivalent model are very close and differ slightly from the results of the full model. However, the difference between full model and local/equivalent model for the sandwich plate with soft cores is smaller than that for the sandwich plate with normal cores. Some local modes can be found in the first 8 modal shapes, such as, the fourth, sixth and eighth order modal shapes of local

![Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets $E_{11}$.](image)

(a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.

Fig. 12. Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets $E_{11}$. (a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.
model and equivalent model, the fourth, sixth and seventh order modal shapes of the full model.

3.2. Sensitivity analysis of the composite sandwich plate

The composite sandwich plate employed in this numerical example as that employed in 3.1 has the same geometry characteristic, material properties and boundary conditions. The stacking sequence of the upper and lower facesheets is [0/90/0]s.

Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets and honeycomb are shown in Figs. 9–17. For the displacements sensitivity coefficients, TSUF and TSLF denote the top surfaces of the upper and lower facesheets respectively, MSUF and MSLF denote the middle surfaces of the upper and lower facesheets respectively, and BSUF and BSLF denote the bottom surfaces of the upper and lower facesheets respectively. For the stresses sensitivity coefficients, TSUF and TSLF denote the top

![Figures](a), (b), (c), (d)

Fig. 13. Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets $E_{22}$. (a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.
surfaces of the first layer of the upper and lower facesheets respectively, MSUF and MSLF denote the top surfaces of the third layer of the upper and lower facesheets respectively, and BSUF and BSLF denote the top surfaces of the fifth layer of the upper and lower facesheets respectively.

The displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets $E_{11}$ and $E_{22}$ are shown in Figs. 12 and 13 respectively. On the basis of the curves in Figs. 12 and 13, the following conclusions can be drawn:

1. The central line distributions of the displacements and the stresses sensitivity coefficients with respect to the parameter $E_{11}$ are similar to those with respect to the parameter $E_{22}$. Besides the
displacements \( u \) and \( v \) in the upper facesheet, the displacements and the stresses of the composite sandwich plates are more sensitive to the parameter \( E_{22} \) than the parameter \( E_{11} \).

2. The sensitivity coefficients of the displacement \( w \) with respect to the parameters \( E_{22} \) and \( E_{11} \) are negative, which means that the out-plane stiffness of the composite sandwich plates increases as the \( E_{11} \) and \( E_{22} \) are increased.

3. The displacements and stresses of the upper facesheet are more sensitive to the parameters \( E_{11} \) and \( E_{22} \) than those of the lower facesheet. The sensitivity coefficients of the displacements \( u \) and \( v \) at the top and bottom surfaces are greater than those at the middle surface in the area around the OFD central point of the upper facesheet, while in the lower facesheet the sensitivity coefficients of the displacements \( u \) and \( v \) decrease along

Fig. 15. Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets \( G_{23} \). (a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.
the thickness direction. However, in the upper and lower facesheets the sensitivity coefficients of the displacement $w$ change just slightly along the thickness direction.

4. In the thickness direction, the sensitivity coefficients of the stresses $\sigma_{11}, \sigma_{22}$ change rapidly especially in the area around the central point, while the sensitivity coefficients of the stress $\sigma_{33}$ change just slightly. The maximum values of sensitivity coefficient of the stresses $\sigma_{11}, \sigma_{22}$ and $\sigma_{33}$ appear at the central point of the upper and lower facesheets. For the displacements, the discussion about the maximum values of sensitivity coefficients with respect to parameters $E_{11}$ can be found in Section 3.1.2.

5. The concentration effect of the stresses sensitivity coefficients with respect to the parameters $E_{22}$ and $E_{11}$ in the upper face-sheet is more significant than that in the lower face-sheet.

The displacements and the stresses sensitivity coefficients with respect to the design variables of the facesheets $G_{12}$ and $G_{23}$ are

![Fig. 16. Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets $t$. (a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.](image-url)
shown in Figs. 14 and 15 respectively. On the basis of the curves in Figs. 14 and 15, the following conclusions can be drawn:

1. The displacements and the stresses in the facesheets of the composite sandwich plates are more sensitive to the parameter $G_{23}$ than to the parameters $G_{12}$. The displacements and stresses of the upper facesheet are more sensitive to the parameters $G_{12}$ and $G_{23}$ than those of the lower facesheet.

2. The sensitivity coefficients of the displacement $w$ with respect to the parameters $G_{12}$ and $G_{23}$ are negative, which means that the out-plane stiffness of the composite sandwich plates increases as the $G_{12}$ and $G_{23}$ are increased.

3. With respect to $G_{12}$, the maximum values of sensitivity coefficients of the displacement $w$ and the stresses $r_{11}$, $r_{22}$ and $r_{33}$ appear at the central point of the upper and lower facesheets. In the upper facesheet the displacements $u$ and $v$ in the area nearby the central point are most sensitive to $G_{12}$.
In the lower facesheet the displacements $u$ at the edges ($x = 0$, 42 mm, $y = 0$ mm) are most sensitive to $G_{12}$, while the displacements $v$ at the edges ($y = 0$, 44 mm, $x = 21$ mm) are most sensitive. With respect to $G_{23}$, besides the displacement $w$ and the stresses of the upper facesheets, the maximum values of sensitivity coefficients appear in the area nearby the central point.

4. For the concentration effect of the sensitivity coefficients with respect to the parameters $G_{23}$, the difference between the upper and lower facesheets is not obvious, while there is very significant difference for the parameters $G_{12}$.

The displacements and the stresses sensitivity coefficients with respect to the parameter of the facesheets $t_s$ (the total thickness of...
1. The displacements and the stresses in the facesheets of the composite sandwich plates are more sensitive to the parameter \( t \) than to the parameter \( h \). And the displacements and stresses of the upper facesheet are more sensitive to the parameters \( t_s \) and \( h \) than those of the lower facesheet.

2. The sensitivity coefficients of the displacement \( w \) with respect to the parameter \( t_s \) are negative, which means that the out-of-plane stiffness of the composite sandwich plates increases rapidly as the \( t_s \) is increased.

3. With respect to the parameter \( t_s \), the central line distributions of the displacements and the stresses sensitivity coefficients are similar to that with respect to the parameter \( E_{11} \). However, the central line distributions of the displacements and the stresses sensitivity coefficients with respect to \( h \) are very complex.

Fig. 19. Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the facesheets \( v \). (a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.
4. The concentration effect of the stresses sensitivity coefficients with respect to the parameters $t_s$ and $h$ in the upper facesheet is more significant than that in the lower facesheet.

It can be seen from Figs. 12–17 that for the material properties of the facesheets the displacements and the stresses of the composite sandwich plates are most sensitive to $G_{23}$. Among all the parameters of the facesheets researched in this paper, the sensitivity values with respect to the thickness $t_s$ are considerably larger than those with respect to the other parameters especially the material properties.

Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the parameter of honeycomb $E_1$ and $v$ are shown in Figs. 18 and 19 respectively. On the basis of the curves in Figs. 18 and 19, the following conclusions can be drawn:

Fig. 20. Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the design variable of the honeycomb $t_c$. (a) Displacements coefficient of the upper facesheet; (b) displacements coefficient of the lower facesheet; (c) stresses coefficient of the upper facesheet; and (d) stresses coefficient of the lower facesheet.
1. The displacements and the stresses in the facesheets of the composite sandwich plates are more sensitive to the parameters \( v \) than the parameters \( E_t \). And the displacements and stresses of the upper facesheet are more sensitive to the parameters \( v \) and \( E_t \) than those of the lower facesheet.

2. The out-plane stiffness of the composite sandwich plates increases rapidly as the \( E_t \) increases while for \( v \) it is just opposite.

3. The maximum values of sensitivity coefficients of the displacements \( u \) and \( v \) with respect to the parameters \( v \) and \( E_t \) appear in the area nearby the central point, while for the displacement \( w \) and the stresses the maximum values of sensitivity coefficients appear just at the central point.

4. For the concentration effect of the sensitivity coefficients with respect to the parameters \( t_c \), the difference between the upper and lower facesheet is not obvious.

Central line distributions of the displacements and the stresses sensitivity coefficients with respect to the parameter \( t_c \) are shown in Fig. 20. On the basis of the curves in Fig. 20, the following conclusions can be drawn:

1. The effects of the parameter \( t_c \) on the displacements \( u, v \) and the stresses \( \sigma_{11}, \sigma_{22} \) in the upper and lower facesheets of the composite sandwich plates are very close while the effects on the displacement \( w \) and stress \( \sigma_{33} \) of the lower facesheet are more obvious than on those of the upper facesheet.

2. The out-plane stiffness of the composite sandwich plates increases rapidly as the \( t_c \) is increased.

3. The maximum values of sensitivity coefficients of the displacements \( u \) and \( v \) with respect to the parameters \( t_c \) appear in the area nearby the central point, while for the displacement \( w \) and the stresses the maximum values of sensitivity coefficients appear just at the central point.

4. For the concentration effect of the sensitivity coefficients with respect to the parameters \( t_c \), the difference between the upper and lower facesheet is not obvious.

### Table 6
Maximum displacement sensitivity coefficients of the honeycomb and the facesheets.

<table>
<thead>
<tr>
<th>( j )</th>
<th>Layer</th>
<th>Honeycomb</th>
<th>Facesheets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z = 1 ) mm</td>
<td>( z = 3 ) mm</td>
<td>( z = 5 ) mm</td>
</tr>
<tr>
<td>( E_{11} )</td>
<td>( u/10^{11} )</td>
<td>( v/10^{12} )</td>
<td>( w/10^{11} )</td>
</tr>
<tr>
<td></td>
<td>0.78731</td>
<td>0.34051</td>
<td>0.41337</td>
</tr>
<tr>
<td></td>
<td>0.47489</td>
<td>0.73583</td>
<td>1.06502</td>
</tr>
<tr>
<td></td>
<td>0.47489</td>
<td>0.73583</td>
<td>1.06502</td>
</tr>
<tr>
<td>( G_{12} )</td>
<td>( v/10^{10} )</td>
<td>( w/10^{10} )</td>
<td>0.88871</td>
</tr>
<tr>
<td></td>
<td>0.90009</td>
<td>0.30858</td>
<td>0.26708</td>
</tr>
<tr>
<td>( G_{23} )</td>
<td>( w/10^{10} )</td>
<td>0.70971</td>
<td>0.85459</td>
</tr>
<tr>
<td></td>
<td>1.78851</td>
<td>0.57286</td>
<td>0.79909</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( w/10^{8} )</td>
<td>2.95267</td>
<td>1.79392</td>
</tr>
<tr>
<td>( t_c )</td>
<td>( w/10^{5} )</td>
<td>3.94229</td>
<td>3.53013</td>
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<tr>
<td></td>
<td>2.57379</td>
<td>1.69192</td>
<td>3.74984</td>
</tr>
<tr>
<td></td>
<td>3.49772</td>
<td>1.10102</td>
<td>1.94882</td>
</tr>
<tr>
<td></td>
<td>(-2.24328)</td>
<td>(-2.36455)</td>
<td>(-2.53448)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{11} )</td>
<td>1</td>
</tr>
<tr>
<td>( E_{22} )</td>
<td>2.82208</td>
</tr>
<tr>
<td>( E_{12} )</td>
<td>4.55845</td>
</tr>
<tr>
<td>( G_{12} )</td>
<td>2.22580</td>
</tr>
<tr>
<td>( \rho_1 )</td>
<td>3.04085</td>
</tr>
<tr>
<td>( \rho_2 )</td>
<td>(-6.90364)</td>
</tr>
<tr>
<td>( t/10^3 )</td>
<td>(-3.71042)</td>
</tr>
</tbody>
</table>

### Table 7
Sensitivity coefficients of the first 4 natural frequencies with respect to the parameters of the facesheets and the honeycomb.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Mode number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{11} )</td>
<td>0.92372</td>
</tr>
<tr>
<td>( v/10^4 )</td>
<td>(-1.00204)</td>
</tr>
<tr>
<td>( \rho_1 )</td>
<td>(-2.23632)</td>
</tr>
<tr>
<td>( t/10^3 )</td>
<td>2.76046</td>
</tr>
</tbody>
</table>
For the parameters of the honeycomb studied in this paper, the concentration effect of the displacements and stresses sensitivity coefficients in the upper and lower facesheets has no significant difference (see Table 6).

The maximum displacements sensitivity coefficients of the honeycomb and the facesheets are listed in Table 3. Based on Table 3, the following conclusions can be drawn:

1. The sensitivity values of the displacements with respect to shape parameters are considerably larger than those with respect to the material properties except the Poisson’s ratio of the honeycomb. In other words, the displacements of the composite sandwich plates are very sensitive to the Poisson’s ratio of the honeycomb.
2. The displacements $u$ and $v$ of the top and bottom surfaces of the honeycomb are more sensitive than those of the middle surface with respect to the parameters of the facesheets, while for the parameters of the honeycomb it is just opposite.
3. The displacement $w$ of the top surface of the honeycomb is more sensitive than that of the bottom surface with respect to the parameters of the facesheets and the Young’s modulus of the honeycomb $E_v$, while for the Poisson’s ratio and the thickness of the honeycomb it is just opposite. Besides Poisson’s ratio of the honeycomb $v$, the displacement $w$ increases as the material properties of the facesheets and the honeycomb are increased, which means that the stiffness of the composite sandwich plates decreases as the Poisson’s ratio of the honeycomb $v$ is increased while the other material properties is just the opposite.
4. The influence of the parameters of the facesheets and the Young’s modulus of the honeycomb is most significant on the displacements of the upper face-sheet, and slightest on the displacements of the lower face-sheet. The influence of the thickness of the honeycomb on the displacements of the honeycomb is most significant, and slightest on the displacements of the upper face-sheet. The influence of the Poisson’s ratio of the honeycomb on the displacements of the honeycomb and the upper face-sheet is more significant than that on the lower face-sheet.

The sensitivity coefficients of the first 4 natural frequencies with respect to the parameters of the facesheets and the honeycomb are listed in Table 7. Based on Table 7, the following conclusions can be drawn:

1. The sensitivity coefficients of the natural frequencies increase as the mode number is increased. The sensitivity values of the natural frequencies with respect to shape parameters are considerably larger than those with respect to the material properties except the Poisson’s ratio of the honeycomb.
2. Besides Poisson’s ratio of the honeycomb $v$, the first 4 natural frequencies increase as the material properties of the composite sandwich plates are increased, since the stiffness of the composite sandwich plates decreases as the Poisson’s ratio of the honeycomb $v$ is increased while the other material properties are just opposite. The increase of the density of the facesheets and the honeycomb makes the natural frequencies decrease.
3. The increase of the thickness of the facesheets makes the natural frequencies decrease, while the increase of the thickness of the honeycomb makes the natural frequencies increase. The increase of the thickness of the facesheets and honeycomb not only improves the stiffness of the whole composite sandwich structure but also enhances the mass. We can find from the sensitivity coefficients that the contribution of the thickness of the facesheets to the mass matrix is greater than to the stiffness matrix, while the contribution of the thickness of the honeycomb is just the opposite.

4. Conclusion remarks

In this paper, a LW/SE method, the SAM and the OFD are employed to studied the sensitivity analysis problems of the composite sandwich plates for the static response and free vibration analysis. From the research, following conclusions are obtained:

1. The present sensitivity analysis schemes with the SAM and the OFD are effective with good convergence and stability. The step size required in the sensitivity analysis based on the OFD is smaller than that required in the sensitivity analysis based on the SAM.
2. Every time the step sizes of the SAM and the OFD reduce one order of magnitude, the relative errors of the displacements and the fundamental frequency sensitivity coefficients also reduce about one order of magnitude. The truncation error is more remarkable than around-off error for the large step size while for the small step size the around-off error plays a key role.
3. For static response sensitivity analysis, the sensitivity analysis schemes based on the full model and the local model can obtain the local concentration effect of the displacements sensitivity coefficients especially in the upper face-sheet, but the local model obviously reduces the computational cost. For the free vibration sensitivity analysis, the sensitivity coefficients of the natural frequencies calculated by the local model and the equivalent model are very close and differ slightly from the results of the full model.
4. For the parameters of the facesheets, the concentration effect of the displacements and stresses sensitivity coefficients in the upper face-sheet is more significant than that in the lower face-sheet. However, for the parameter $G_{12}$ of the honeycomb and the facesheets, the concentration effects of the displacements and stresses sensitivity coefficients in the upper and lower facesheets have no significant difference.
5. The sensitivity values of the displacements, stresses and natural frequencies with respect to shape parameters are considerably larger than those with respect to the material properties except the Poisson’s ratio of the honeycomb.

References


