瑞利-里兹法
Rayleigh-Ritz Analysis

Tuesday, March 9, 2010

瑞利-里兹法

- 瑞利商一般仅用于求解系统的第一阶振型—精度取决于假设振型对第一阶主振型的近似程度
- 如何假设合理的主振型？
  - 里兹法—取为一组线性独立的向量的线性组合

$$\bar{\phi} = \sum_{j=1}^{q} x_j \boldsymbol{\psi}_j = \boldsymbol{\Psi x} \qquad \boldsymbol{\Psi} = [\psi_1 \quad \psi_2 \quad \cdots \quad \psi_q]$$

$$\boldsymbol{x} = [x_1 \quad x_2 \quad \cdots \quad x_q]^{\mathrm{T}}$$

瑞利商 $\rho(\bar{\phi}) = \dfrac{\bar{\phi}^T \boldsymbol{K} \bar{\phi}}{\bar{\phi}^T \boldsymbol{M} \bar{\phi}}$

$$\Downarrow$$

$$\rho(\bar{\phi}) = \frac{\boldsymbol{x}^T \bar{\boldsymbol{K}} \boldsymbol{x}}{\boldsymbol{x}^T \bar{\boldsymbol{M}} \boldsymbol{x}} \qquad \bar{\boldsymbol{K}} = \boldsymbol{\Psi}^T \boldsymbol{K \Psi} \qquad \bar{\boldsymbol{M}} = \boldsymbol{\Psi}^T \boldsymbol{M \Psi}$$

瑞利-里兹法

$$\rho(\bar{\phi}) = \frac{\boldsymbol{x}^T \bar{\boldsymbol{K}} \boldsymbol{x}}{\boldsymbol{x}^T \bar{\boldsymbol{M}} \boldsymbol{x}}$$

瑞利商在真实主振型处取驻值：

$$\frac{\partial \rho(\bar{\phi})}{\partial \boldsymbol{x}} = \frac{\bar{\boldsymbol{K}}\boldsymbol{x}(\boldsymbol{x}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{x}) - \bar{\boldsymbol{M}}\boldsymbol{x}(\boldsymbol{x}^{\mathrm{T}}\bar{\boldsymbol{K}}\boldsymbol{x})}{(\boldsymbol{x}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{x})^2} = 0$$

$$\bar{\boldsymbol{K}}\boldsymbol{x}(\boldsymbol{x}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{x}) - \bar{\boldsymbol{M}}\boldsymbol{x}(\boldsymbol{x}^{\mathrm{T}}\bar{\boldsymbol{K}}\boldsymbol{x}) = 0$$

$$\bar{\boldsymbol{K}}\boldsymbol{x} = \rho \bar{\boldsymbol{M}}\boldsymbol{x} \Rightarrow \rho_i, \boldsymbol{x}_i = [x_1^i \; x_2^i \cdots x_q^i]^T, \; i = 1, 2, \cdots, q \le n$$

$$\lambda_i \approx \rho_i$$

$$\boldsymbol{\phi}_i \approx \bar{\boldsymbol{\phi}}_i = \boldsymbol{\Psi x}_i = \sum_{j=1}^{q} x_j^i \boldsymbol{\psi}_j$$

解的上限性

若假设的振型与第一阶主振型正交：$a_1 = 0$

$$\boldsymbol{K\phi} = \lambda \boldsymbol{M\phi} \quad \rho(\phi) = \lambda = \frac{\phi^T \boldsymbol{K} \phi}{\phi^T \boldsymbol{M} \phi} = \frac{\sum_{j=1}^{n} a_j^2 \omega_j^2}{\sum_{j=1}^{n} a_j^2} \Rightarrow \omega_2^2 \le \rho(\phi) \le \omega_n^2$$

$$\Downarrow$$

$$\lambda_2 = \min \rho(\boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} : \boldsymbol{\phi} \in V_n, \; \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\phi}_1 = 0$$

$$\bar{\boldsymbol{K}}\boldsymbol{x} = \rho \bar{\boldsymbol{M}}\boldsymbol{x} \qquad \bar{\boldsymbol{\phi}} = \boldsymbol{\Psi x} \Rightarrow (\rho_1, \bar{\boldsymbol{\phi}}_1); (\rho_2, \bar{\boldsymbol{\phi}}_2); \cdots; (\rho_q, \bar{\boldsymbol{\phi}}_q)$$

$$\rho_2 = \min \rho(\bar{\boldsymbol{\phi}}), \quad \forall \bar{\boldsymbol{\phi}} : \bar{\boldsymbol{\phi}} \in V_q, \; \bar{\boldsymbol{\phi}}^{\mathrm{T}} \boldsymbol{M} \bar{\boldsymbol{\phi}}_1 = 0$$

$$\tilde{\rho}_2 = \min \rho(\bar{\boldsymbol{\phi}}), \quad \forall \bar{\boldsymbol{\phi}} : \bar{\boldsymbol{\phi}} \in V_q, \; \bar{\boldsymbol{\phi}}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\phi}_1 = 0$$

$$V_q \subseteq V_n \Rightarrow \lambda_2 \le \tilde{\rho}_2$$

$$\bar{\boldsymbol{\phi}}_1 \in V_q, \boldsymbol{\phi}_1 \in V_n \Rightarrow \tilde{\rho}_2 \le \rho_2$$

$$\Rightarrow \lambda_2 \le \rho_2$$

$$\lambda_i \le \rho_i, \; i = 1, 2, \cdots, q$$

瑞利-里兹法求得的近似特征值是精确值的上限！

讨论

- 高阶近似特征值的精度低于低阶近似特征值，故一般取$q = 2p$；

$$\bar{\boldsymbol{\phi}}^{\mathrm{T}} \boldsymbol{M} \bar{\boldsymbol{\phi}}_j = 0, \; j = 1, 2, \cdots, i-1$$

- 如果每个里兹基向量都可以表示成系统前q阶主振型的线性组合，即：

$$\boldsymbol{\psi}_j = \boldsymbol{\Phi y}^j = \sum_{k=1}^{q} y_k^j \boldsymbol{\phi}_k \; (j = 1, 2, \cdots, q) \qquad \boldsymbol{\Psi} = \boldsymbol{\Phi Y}$$

$$\bar{\boldsymbol{K}}\boldsymbol{x} = \rho \bar{\boldsymbol{M}}\boldsymbol{x} \Leftarrow \bar{\boldsymbol{K}} = \boldsymbol{\Psi}^T \boldsymbol{K \Psi} = \boldsymbol{Y}^T \boldsymbol{\Lambda Y}$$

$$\bar{\boldsymbol{M}} = \boldsymbol{\Psi}^T \boldsymbol{M \Psi} = \boldsymbol{Y}^T \boldsymbol{Y}$$

$$\Downarrow$$

$$\boldsymbol{Y}^{\mathrm{T}}(\boldsymbol{\Lambda} - \rho \boldsymbol{I})\boldsymbol{z} = 0 \qquad \boldsymbol{z} = \boldsymbol{Yx}$$

$$\left|\boldsymbol{Y}^{\mathrm{T}}(\boldsymbol{\Lambda} - \rho \boldsymbol{I})\right| = \left|\boldsymbol{Y}^{\mathrm{T}}\right| \left|\boldsymbol{\Lambda} - \rho \boldsymbol{I}\right| = 0$$

$$\left|\boldsymbol{\Lambda} - \rho \boldsymbol{I}\right| = 0 \Rightarrow \rho_i = \lambda_i$$

里兹法得到精确解！

讨论

- 只要子空间V_q接近于V_n，里兹法就能得到很好的近似解 — 比分别选取q个接近于真实主振型的假设振型容易的多！
- 可取$\boldsymbol{K\Psi} = \boldsymbol{R}$为里兹基；

例3-5

用瑞利—里兹法求解广义特征值问题$K\phi=\lambda M\phi$的近似解，其中

$$K=\begin{bmatrix}2&-1&0\\-1&4&-1\\0&-1&2\end{bmatrix};\quad M=\begin{bmatrix}\frac{1}{2}&0&0\\0&1&0\\1&0&\frac{1}{2}\end{bmatrix}$$

本问题的精确解为：$\lambda_1=2,\lambda_2=4,\lambda_3=6$

例3-5

$$R=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\quad \Psi=K^{-1}R=\begin{bmatrix}\frac{7}{12}&\frac{1}{12}\\\frac{1}{6}&\frac{1}{6}\\\frac{1}{12}&\frac{7}{12}\end{bmatrix}$$

$$\bar{K}=\Psi^{\mathrm{T}}K\Psi=\frac{1}{12}\begin{bmatrix}7&1\\1&7\end{bmatrix}$$

$$\bar{M}=\Psi^{\mathrm{T}}M\Psi=\frac{1}{144}\begin{bmatrix}29&11\\11&29\end{bmatrix}$$

$$\bar{K}x=\rho\bar{M}x \Rightarrow (\rho_1,x_1)=\left(2.4004,\begin{bmatrix}1.3418\\1.3418\end{bmatrix}\right)$$

$$(\rho_2,x_2)=\left(4.0032,\begin{bmatrix}2.0008\\-2.0008\end{bmatrix}\right)$$

例3-5

$$R=\begin{bmatrix}1&0\\1&1\\1&0\end{bmatrix}\quad \Psi=\begin{bmatrix}\frac{5}{6}&\frac{1}{6}\\\frac{2}{3}&\frac{1}{3}\\\frac{5}{6}&\frac{1}{6}\end{bmatrix}$$

瑞利—里兹法的结果完全依赖于里兹向量的选取！

$$\bar{K}=\begin{bmatrix}\frac{7}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix};\quad \bar{M}=\frac{1}{36}\begin{bmatrix}41&13\\13&5\end{bmatrix}$$

$$\bar{K}x=\rho\bar{M}x \Rightarrow (\rho_1,x_1)=\left(2.00,\begin{bmatrix}0.70711\\0.70711\end{bmatrix}\right)$$

$$(\rho_2,x_2)=\left(6.00,\begin{bmatrix}-2.1213\\6.3640\end{bmatrix}\right)$$

讨论

- 瑞利-里兹法将n阶特征值问题缩减为$q(q\ll n)$阶特征值问题 $\Longrightarrow$ 可用广义雅可比法高效求解；
- 瑞利-里兹法的结果取决于里兹向量的选取 $\Longrightarrow$ 如何提高瑞利-里兹法解的精度？

如何高效求解大型广义特征值问题？