复模态理论

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复模态理论

如何对非比例阻尼系统的运动方程进行解耦？

$$M\ddot{x}+C\dot{x}+Kx=f(t)$$
$$M\dot{x}-M\dot{x}=0$$

$$y(t)=\begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix}+\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}=\begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

$A\dot{y}(t)-By(t)=\tilde{f}(t)$ —状态方程

$$A=\begin{bmatrix} C & M \\ M & 0 \end{bmatrix},\quad B=\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$$

复模态理论

自由振动

$$A\dot{y}(t)-By(t)=0 \Leftarrow y=\phi e^{\lambda t}$$

$$\lambda A\phi-B\phi=0 \Rightarrow \Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_{2n}),\quad \Phi=[\ \phi_1 \quad \phi_2 \quad \cdots \quad \phi_{2n}\ ]$$

$$A\Phi\Lambda-B\Phi=0$$

n对共轭复特征对

$$\lambda_i A\phi_i-B\phi_i=0 \Rightarrow \lambda_i\phi_i^{\mathrm{T}}A^{\mathrm{T}}\phi_j-\phi_i^{\mathrm{T}}B^{\mathrm{T}}\phi_j=0$$

$$\lambda_j A\phi_j-B\phi_j=0 \Rightarrow \lambda_j\phi_i^{\mathrm{T}}A\phi_j-\phi_i^{\mathrm{T}}B\phi_j=0$$

若M、C、K均对称

$$\phi_i^{\mathrm{T}}A\phi_j=a_j\delta_{ij} \Leftarrow (\lambda_i-\lambda_j)\phi_i^{\mathrm{T}}A\phi_j=0$$

$$\phi_i^{\mathrm{T}}B\phi_j=b_j\delta_{ij}$$

$$\Phi^{\mathrm{T}}A\Phi=a$$

$$\Phi^{\mathrm{T}}B\Phi=b$$

$\lambda_i=\dfrac{b_i}{a_i}$ — b_i 复模态刚度，a_i 复模态质量

复模态理论

将状态向量在由复振型矩阵的各列向量构成的$2n$维复模态空间中展开

$y(t)=\Phi q$　　q — $2n$维复模态坐标向量

$$A\dot{y}(t)-By(t)=0$$

$\dot{q}-\Lambda q=0$　在$2n$维复模态空间中，模态方程已解耦

$$\dot{q}_i-\lambda_i q_i=0,\ i=1,2,\cdots,2n$$

$$q_i(t)=q_i(0)e^{\lambda_i t}$$

$$q=dq_0 \qquad d=\mathrm{diag}(e^{\lambda_1 t},e^{\lambda_2 t},\cdots,e^{\lambda_{2n} t})$$

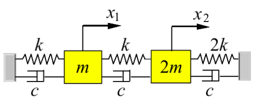
$$q_0=a^{-1}\Phi^{\mathrm{T}}Ay(0)$$

$$y(t)=\Phi dq_0$$

例5-1

求图示系统在初始条件$x_0=[0 \quad 0]^T$　$\dot{x}_0=[0 \quad 1]^T$，下的响应。

$k=m=c=1$



$$M\ddot{x}+C\dot{x}+Kx=0$$

$$M=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ C=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ K=\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\Lambda=\begin{bmatrix} 1.0 & 0 \\ 0 & 2.5 \end{bmatrix},\ \Phi=\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \qquad \Phi^T C\Phi=\begin{bmatrix} 2 & -1 \\ -1 & 14 \end{bmatrix}$$

不能解耦！

例5-1

$$A=\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}=\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}=\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$(\lambda A-B)\phi=0$$

$$y(0)=\begin{Bmatrix} x(0) \\ \dot{x}(0) \end{Bmatrix}=[0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}}$$

例5-1

$$\Lambda = \mathrm{diag}(-1.1727+1.0321i, -1.1727-1.0321i, -0.3273+0.9578i, -0.3273-0.9578i)$$

$$\Phi = \begin{bmatrix} -0.4754-0.3428i & -0.4754+0.3428i & -0.4841-0.3440i & -0.4841+0.3440i \\ 0.1878+0.0707i & 0.1878-0.0707i & -0.4744-0.5256i & -0.4744+0.5256i \\ 0.9113-0.0887i & 0.9113+0.0887i & 0.4879-0.3510i & 0.4879+0.3510i \\ -0.2933+0.1109i & -0.2933-0.1109i & 0.6586-0.2824i & 0.6586+0.2824i \end{bmatrix}$$

$$\boldsymbol{a} = \Phi^{\mathrm{T}}\boldsymbol{A}\Phi$$

$$= \mathrm{diag}(-0.7712+0.3610i,\ -0.7712-0.3610i,\ -2.5255-0.0166i,\ -2.5255+0.0166i)$$

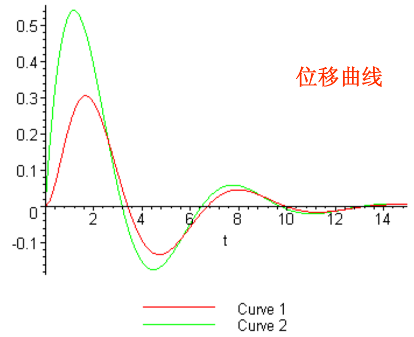
$$\boldsymbol{b} = \Phi^{\mathrm{T}}\boldsymbol{B}\Phi$$

$$= \mathrm{diag}(0.5320-1.2193i,\ 0.5320+1.2193i,\ 0.8423-2.4134i,\ 0.8423+2.4134i)$$

$$\boldsymbol{q}_0 = \boldsymbol{a}^{-1}\Phi^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y}(0) = \begin{bmatrix} -0.3291-0.3374i \\ -0.3291+0.3374i \\ 0.3784+0.4138i \\ 0.3784-0.4138i \end{bmatrix}$$

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}\boldsymbol{d}\boldsymbol{q}_0$$

例5-1



例5-1