Elasto-plastic analysis based on collocation with moving least square method*

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Abstract

A meshless approach based on Moving Least Square Method is developed for the elasto-plastic analysis, in which the incremental formulation is used. In this approach, the displacement shape functions are constructed by using the moving least square approximation, and the discrete governing equations of elasto-plastic material are constructed by using direct collocation method. The boundary conditions are also imposed by collocation. The method established is a truly meshless method, as it does not need any mesh, either for purpose of interpolation of the solution variables, or for purpose of construction of the discrete equations. It is simply formulated and very efficient, and no post-processing procedure is required to compute the derivatives of the unknown variables, since the solution from this method based on the moving least square approximation is already smooth enough. Numerical examples are given to verify the accuracy of the proposed meshless method for elasto-plastic analysis.

1 Introduction

During recent years, much attention in computational mechanics has been paid to the development of meshless methods. In these methods, the domain of interest is described by a set of scattered points. Every point has an associated sub-domain and a local approximation is achieved. Thus, in these meshless methods, the establishment of shape functions does not require mesh because the interpolation of field variables is accomplished at the global level and built with the information provided by points in its sub-domain.

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In the finite element method, the elements are used to discrete the domain of the problem. In each element, approximation polynomials are used for the test and trial functions. The integrals in the weak form are evaluated by dividing the domains into subdomains which correspond exactly to the elements, and the quadrature scheme chosed is employed over each element subdomain.

Meshless methods offer considerable promise in areas where the traditional finite element approximations have encountered difficulties due to the need to continually remesh the domain. These methods can avoid the difficult task of generating 3D mesh, and are ideal for problems which need remeshing, such as adaptivity, fracture problems, and large deformation problems. Several meshless methods have been developed, including Smooth Particle Hydrodynamics(SPH)[1], Diffuse Element Method(DEM)[2], Element Free Galerkin Method(EFG)[3], Reproducing Kernel Particle Method(RKPM)[4], Finite Point Method(FP)[5], hp Clouds Method(HP)[6], Meshless Local Petrov-Galerkin method(MLPG)[7]–[8], Local Boundary Integral Equation method(LBIE)[9]–[10], Radial Basis Functions Method (RBF)[11], and several others[12].

In these meshless methods, there are mainly four different ways to establish approximation, namely, smooth particle hydrodynamics method (SPH), moving least square method(MLS), partition of unity method, and radial basis function. They construct approximation entirely in terms of points.

To construct the discrete equations, two methods are mainly used in existing meshless methods, collocation method and Galerkin method. For Galekin-based meshless methods, two major difficulties exist. First, the imposition of essential boundary conditions is quite awkward because of the non-interpolatory character of the meshless approximation, and several approaches have been developed [13]–[15]. Second, background mesh is always required in these methods for purpose of calculating the quadrature of the weak form, so they are not truly meshless methods. In contrast, collocation-based methods are truly meshless methods, and they are very simple and efficient.

Although the first meshless method was developed years ago, the more important theoretical research works and their applications in computational mechanics have been recently performed, and many research works are only starting. In this paper, a meshless approach for elasto-plastic problem is presented based on the moving least square method and direct collocation, in which the incremental formulation is used. The increments of the displacement are constructed by using the MLS approximation, and the governing equations of the elasto-plastic material are discretized by the direct collocation method. The incremental boundary conditions are also imposed by collocation. Not needing any mesh, either for purpose of interpolation of the solution variables, or for purpose of construction of the discrete equations, the method established here is a truly meshless method. It is simply formulated and very efficient, and no post-processing procedure is required. Numerical examples are given to verify the accuracy of the proposed meshless method for elasto-plastic analysis.

2 MLS Approximation

In moving least square approximation scheme, the function $u(\mathbf{x})$ in domain Ω is approximated by $u^h(\mathbf{x})$, as

$$u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \cdot \mathbf{a}(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x}) \cdot a_i(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega \tag{1}$$

where Ω is the domain of definition of $\mathbf{x}$, $\mathbf{a}(\mathbf{x})$ is the vector containing coefficients $a_i(\mathbf{x})$, which are functions of the spatial coordinates. $\mathbf{p}(\mathbf{x})$ is vector of basis containing complete monomial functions, and m is the number of terms in the basis, such as $m = 6$ in 2D:

$$\mathbf{p}^T(\mathbf{x}) = [1, x, y, x^2, xy, y^2] \tag{2}$$

It is also possible to use any other functions in the basis. For example, in problems with singular solutions, singular functions can be included.

The coefficients $a_i(\mathbf{x})$ is determined by minimizing a weighted discrete L_2 norm,

$$R = \sum_{I=1}^{N} w_I(\mathbf{x}) \cdot [\mathbf{p}^T(\mathbf{x}_I) \cdot \mathbf{a}(\mathbf{x}) - u_I]^2 \tag{3}$$

where N is the total number of nodes scattered in the domain Ω to construct the approximation of the function $u(\mathbf{x})$, $w_I(\mathbf{x})$ is the weight function associated with node $\mathbf{x}_I$, and u_I is the value of function $u(\mathbf{x})$ at $\mathbf{x}_I$.

Solving $\mathbf{a}(\mathbf{x})$ by minimizing R in (3) and substitute it into (1) lead to

$$u^h(\mathbf{x}) = \sum_{I=1}^{N} \phi_I(\mathbf{x}) \cdot u_I, \quad \forall \mathbf{x} \in \Omega \tag{4}$$

where,

$$\phi_I(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) \cdot [\mathbf{A}^{-1}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x})]_{jI} \tag{5}$$

$$\mathbf{A}(\mathbf{x}) = \sum_{I=1}^{N} w_I(\mathbf{x}) \cdot \mathbf{p}(\mathbf{x}_I) \cdot \mathbf{p}^T(\mathbf{x}_I) \tag{6}$$

$$\mathbf{B}(\mathbf{x}) = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \cdots w_N(\mathbf{x})\mathbf{p}(\mathbf{x}_N),] \tag{7}$$

Function $\phi_I(\mathbf{x})$ is usually called the shape function of MLS approximation corresponding to node $\mathbf{x}_I$. Note that,

$$\phi_I(\mathbf{x}_J) \neq \delta_{IJ}, \quad u^h(\mathbf{x}_I) \neq u_I \tag{8}$$

This non-interpolatory character makes the imposition of essential boundary conditions difficulty.

The weight function $w_I(\mathbf{x})$ is compactly supported, which takes its maximum value at the node $\mathbf{x}_I$ and vanishes outside a surrounding region, so $\phi_I(\mathbf{x}) = 0$ for $\mathbf{x}$ not in the support of node $\mathbf{x}_I$. Thus, the moving least square approximation possesses local characteristics. That is, for a given point, the approximate value is only based on the information provided by the closest points in its sub-domain.

The commonly used weight functions are the exponential and spline functions with compact support. The exponential weight function corresponding to node $\mathbf{x}_I$ may be written as

$$w_I(\mathbf{x}) = \begin{cases} \exp\left[-(\frac{r_I}{h_I c_I})^2\right] & 0 \leq r_I \leq h_I \\ 0 & r_I > h_I \end{cases} \tag{9}$$

where $r_I = \|\mathbf{x} - \mathbf{x}_I\|$ is the distance from node $\mathbf{x}_I$ to point $\mathbf{x}$, c_I is a constant, and h_I is the size of the support for the weight function $w_I(\mathbf{x})$.

The cubic spline weight function corresponding to node $\mathbf{x}_I$ is

$$w_I(\mathbf{x}) = \begin{cases} \frac{2}{3} - 4(\frac{r_I}{h_I})^2 + 4(\frac{r_I}{h_I})^3 & 0 \le \frac{r_I}{h_I} \le \frac{1}{2} \\ \frac{4}{3} - 4(\frac{r_I}{h_I}) + 4(\frac{r_I}{h_I})^2 - \frac{4}{3}(\frac{r_I}{h_I})^3 & \frac{1}{2} \le \frac{r_I}{h_I} \le 1 \\ 0 & \frac{r_I}{h_I} > 1 \end{cases} \tag{10}$$

Consider a function given by the linear combination of the basis functions, namely

$$u(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x}) \cdot \alpha_i \tag{11}$$

Then if $a_i(\mathbf{x}) = \alpha_i$, R in (3) will vanish and it will be the minimum. Thus

$$u^h(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x}) \cdot \alpha_i = u(\mathbf{x}) \tag{12}$$

so any function in the basis can be reproduced exactly. Therefore, if the basis includes all constant and linear monomials, linear consistency is satisfied; and any singular functions included in the basis can also be reproduced exactly.

3 Governing Equations

For the elasto-plastic problem, the incremental formulation is frequently used[16]. The governing equations for elasto-plastic problem are given by

$$\mathbf{B}(\boldsymbol{\sigma}(\mathbf{x})) - \mathbf{f}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{13}$$

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_{\mathbf{u}} \tag{14}$$

$$\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{x}) = \bar{\mathbf{t}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_{\mathbf{t}} \tag{15}$$

where B is a differential operator, $\mathbf{n}$ is the outward normal direction of boundary $\Gamma_{\mathbf{t}}$, $\mathbf{f}(\mathbf{x})$ is the body force vector, $\bar{\mathbf{u}}(\mathbf{x})$ is the prescribed displacement on $\Gamma_{\mathbf{u}}$, and $\bar{\mathbf{t}}(\mathbf{x})$ is the prescribed traction on $\Gamma_{\mathbf{t}}$.

Using MLS method, the incremental displacement is approximated by

$$\begin{aligned} du(\mathbf{x}) &= \sum_{I=1}^{N} \phi_I(\mathbf{x}) \cdot du_I \\ dv(\mathbf{x}) &= \sum_{I=1}^{N} \phi_I(\mathbf{x}) \cdot dv_I \end{aligned} \tag{16}$$

where $\phi_I(\mathbf{x})$ is the MLS shape function as shown in (5), du_I and dv_I are incremental unknown quantities associated with node $\mathbf{x}_I$, N is the number of the total nodes.

For problem with infinitesimally small displacements and strains, the incremental relation between displacements and strains is linear, namely

$$d\boldsymbol{\varepsilon}(\mathbf{x}) = \left\{ \begin{array}{c} d\varepsilon_x \\ d\varepsilon_y \\ d\gamma_{xy} \end{array} \right\} = \left\{ \begin{array}{l} \sum \frac{\partial \phi_I(\mathbf{x})}{\partial x} \cdot du_I \\ \sum \frac{\partial \phi_I(\mathbf{x})}{\partial y} \cdot dv_I \\ \sum \frac{\partial \phi_I(\mathbf{x})}{\partial x} \cdot dv_I + \sum \frac{\partial \phi_I(\mathbf{x})}{\partial y} \cdot du_I \end{array} \right\} \tag{17}$$

Using the Mises yield condition and isotropic hardening rule, the stress increments can be obtained as

$$d\boldsymbol{\sigma}(\mathbf{x}) = \left\{ \begin{array}{c} d\sigma_x \\ d\sigma_y \\ d\tau_{xy} \end{array} \right\} = \mathbf{D} \cdot d\boldsymbol{\varepsilon}(\mathbf{x}) \tag{18}$$

where

$$\mathbf{D} = \left\{ \begin{array}{ll} \mathbf{D^e} & \textit{in elastic zone} \\ (\mathbf{D^e} - \mathbf{D^p}) & \textit{in plastic zone} \end{array} \right\} \tag{19}$$

and

$$\mathbf{D}^e = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{20}$$

$$\mathbf{D}^p = \frac{E}{B(1-\nu^2)} \begin{bmatrix} (s_x+\nu s_y)^2 & (s_x+\nu s_y)\cdot(s_y+\nu s_x) & (1-\nu)\cdot(s_x+\nu s_y)\cdot\tau_{xy} \\ & (s_y+\nu s_x)^2 & (1-\nu)\cdot(s_y+\nu s_x)\cdot\tau_{xy} \\ \textit{Symmetry} & & (1-\nu)^2\cdot\tau_{xy}^2 \end{bmatrix} \tag{21}$$

in which, E is Young's modulus , v is Poisson's ratio, s_x and s_y are deviate stress, $G = \frac{E}{2(1+\nu)}$, and

$$B = s_x^2 + s_y^2 + 2\nu \cdot s_x \cdot s_y + 2(1-\nu) \cdot \tau_{xy}^2 + \frac{2(1-\nu) \cdot E^p \cdot \sigma_s^2}{9G} \tag{22}$$

in which, E^P is plastic hardening modulus and σ_s is yield stress.

4 Elasto-plastic Properties

In addition to the elastic stress-strain relations, four properties characterize the material behavior in numerical calculation: initial yield condition, harding rule, flow rule, and loading-unloading rule([16]–[17]).

A initial yield condition, which specifies the state of multiaxial stresses corresponding to the start of plastic flow. For isotropic material, Mises initial yield condition is:

$$F^0(\boldsymbol{\sigma}) = \frac{1}{2} s_{ij} s_{ij} - \frac{\sigma_{s0}^2}{3} = 0 \tag{23}$$

in which, σ_{s0} is initial yield stress, s_{ij} is deviate stress.

A harding rule, which specifies how the yield condition is modified during plastic flow. For isotropic hardening material, Mises harding rule is:

$$F(\boldsymbol{\sigma}) = \frac{1}{2} s_{ij} s_{ij} - \frac{\sigma_s^2(\overline{\varepsilon}^p)}{3} = 0 \tag{24}$$

in which, σ_s is yield stress at current time, $\overline{\varepsilon}^p$ is the effective plastic strain which is related to plastic strain increments $d\varepsilon_{ij}^p$ by

$$\overline{\varepsilon}^p = \int d\overline{\varepsilon}^p = \int (\frac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p)^{\frac{1}{2}} \tag{25}$$

A flow rule, which relates the plastic strain increments to the current stresses and the stress increments subsequent to yielding. For isotropic hardening material, the Mises flow rule,

$$f = \frac{1}{2} s_{ij} s_{ij} \tag{26}$$

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{27}$$

$$d\bar{\varepsilon}^{p} = (\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p})^{\frac{1}{2}} = \frac{2}{3} d\lambda \cdot \sigma_s \tag{28}$$

$$d\lambda = \frac{(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \cdot \mathbf{D}^e \cdot d\boldsymbol{\varepsilon}}{(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \cdot \mathbf{D}^e \cdot (\frac{\partial f}{\partial \boldsymbol{\sigma}}) + \frac{4}{9}\sigma_s^2 \cdot E^p} \tag{29}$$

in which $\mathbf{D}^e$ is defined as (20).

A loading-unloading rule, which specifies what will happen from a plastic state, continue plastic loading or elastic unloading . This rule is used to select the true relation between stress increments and strain increments.

If $F = 0$ and $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0$, then continue plastic loading;

If $F = 0$ and $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0$, then elastic unload from plastic state;

If $F = 0$ and $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0$, then for ideal-plastic material, plastic flow will continue. But for material with harden, it will keep plastic state without any new plastic flow.

5 Direct Collocation

In the direct collocation, the discrete equations at loading step $t+\Delta t$ are obtained by imposing governing equations on the nodes used to construct approximation. The governing equations are assumed to be satisfied at loading step t. For the nodes located inside the domain, (13) must be satisfied, and for those located on boundary, (14) or (15) is satisfied. Thus, the solution at loading step $t + \Delta t$ can be obtained without using any mesh.

Since the stress depends nonlinearly on the nodal displacements in plastic state, it is necessary to iterate in the solution of each step associated with plastic. The modified Newton-Raphson iteration is used. For the numerical method, the load level when plastic state begining is first calculated, and this step is completly elastic. After that, plastic will happen, and the solution step with iteration will be introduced.

The following procedure is used to obtain the solution for the plastic-loading step $t+\Delta t$.

(a) Let the starting condition of iteration be

$$\begin{cases} {}^{t+\Delta t}\mathbf{u}^{(0)} = {}^{t}\mathbf{u} \\ {}^{t+\Delta t}\boldsymbol{\varepsilon}^{(0)} = {}^{t}\boldsymbol{\varepsilon} \\ {}^{t+\Delta t}\boldsymbol{\sigma}^{(0)} = {}^{t}\boldsymbol{\sigma} \\ {}^{t+\Delta t}\mathbf{D}^{(0)} =^{t} \mathbf{D} \end{cases} \tag{30}$$

Here, left-superscript t and $t + \Delta t$ specify the loading setp, right-superscript specifies the iterating step of current loading step.

(b) Using MLS method, incremental displacement $\Delta\mathbf{u}^{(i)}$ is defined as (16), with unknown incremental quantities $\Delta {u_I}^{(i)}$and $\Delta {v_I}^{(i)}$, then

$${}^{t+\Delta t}\mathbf{u}^{(i)} =^{t+\Delta t} \mathbf{u}^{(i-1)} + \Delta\mathbf{u}^{(i)} \tag{31}$$

(c) For problem with infinitesimally small displacements and strains, the linear incremental relation between displacement and strains is used. So incremental strains are

$$\Delta\boldsymbol{\varepsilon}^{(i)} = \begin{Bmatrix} \Delta\varepsilon_x^{(i)} \\ \Delta\varepsilon_y^{(i)} \\ \Delta\gamma_{xy}^{(i)} \end{Bmatrix} = \begin{Bmatrix} \sum \frac{\partial\phi_I(\mathbf{x})}{\partial x} \cdot \Delta {u_I}^{(i)} \\ \sum \frac{\partial\phi_I(\mathbf{x})}{\partial y} \cdot \Delta {v_I}^{(i)} \\ \sum \frac{\partial\phi_I(\mathbf{x})}{\partial x} \cdot \Delta v_I^{(i)} + \sum \frac{\partial\phi_I(\mathbf{x})}{\partial y} \cdot \Delta {u_I}^{(i)} \end{Bmatrix} \tag{32}$$

$${}^{t+\Delta t}\boldsymbol{\varepsilon}^{(i)} =^{t+\Delta t} \boldsymbol{\varepsilon}^{(i-1)} + \Delta\boldsymbol{\varepsilon}^{(i)} \tag{33}$$

(d) According the state of iterating step $(i-1)$ and (19-21), ${}^{t+\Delta t}\mathbf{D}^{(i-1)}$ can be calculated by

$${}^{t+\Delta t}\mathbf{D}^{(i-1)} = \begin{Bmatrix} \mathbf{D^e} & for\ elastic\ load/unload \\ (\mathbf{D^e} - \mathbf{D^p}) & for\ plastic\ flow \end{Bmatrix} \tag{34}$$

(e) According to Euler method, using the stress-strain relation of iterating step $(i-1)$, the approximate incremental stress is calculated,

$$\begin{cases} \Delta\boldsymbol{\sigma}^{(i)}={}^{t+\Delta t}\mathbf{D}^{(i-1)}\cdot\Delta\boldsymbol{\varepsilon}^{(i)} \\ {}^{t+\Delta t}\boldsymbol{\sigma}^{(i)}={}^{t+\Delta t}\boldsymbol{\sigma}^{(i-1)}+\Delta\boldsymbol{\sigma}^{(i)} \end{cases} \tag{35}$$

(f) Now, discrete equations can be obtained by using direct collocation

$$\begin{cases} \mathbf{B}({}^{t+\Delta t}\boldsymbol{\sigma}^{(i)})-{}^{t+\Delta t}\mathbf{f}=0, & for\ \mathbf{x}_I\in\Omega \\ {}^{t+\Delta t}\mathbf{u}^{(i)}-{}^{t+\Delta t}\bar{\mathbf{u}}=0, & for\ \mathbf{x}_I\in\Gamma_u \\ \mathbf{n}\cdot{}^{t+\Delta t}\boldsymbol{\sigma}^{(i)}={}^{t+\Delta t}\bar{\mathbf{t}}, & for\ \mathbf{x}_I\in\Gamma_t \end{cases} \tag{36}$$

The unknown incremental quantities can be sloved from these equations, and then, $\Delta\mathbf{u}^{(i)}$and $\Delta\boldsymbol{\varepsilon}^{(i)}$ can be obtained from (16).

(g) Because the $\Delta\boldsymbol{\sigma}^{(i)}$ calculated by step (e) is based on linear-approximate, the incremental relation between stress and strain should be verified as

$$\Delta\boldsymbol{\sigma}^{(i)}=\int_{\varepsilon^{(i-1)}}^{\varepsilon^{(i)}}\mathbf{D}\cdot d\boldsymbol{\varepsilon} \tag{37}$$

The incremental stress and effective plastic strain $\Delta\overline{\varepsilon}^p$ should be re-calculated. Let m identify the portion of incremental strain taken elastically, then

(g.1) Calculate the stress incerment $\Delta\widetilde{\boldsymbol{\sigma}}$, assuming elastic behavior

$$\begin{cases} \Delta\widetilde{\boldsymbol{\sigma}}=\mathbf{D}^e\cdot\Delta\boldsymbol{\varepsilon}^{(i)} \\ \widetilde{\boldsymbol{\sigma}}=\boldsymbol{\sigma}^{(i-1)}+\Delta\widetilde{\boldsymbol{\sigma}} \end{cases} \tag{38}$$

(g.2) Calculate value of the yield-function $F(\boldsymbol{\sigma}^{(i-1)},\overline{\varepsilon}^{p(i-1)})$ and $F(\widetilde{\boldsymbol{\sigma}},\overline{\varepsilon}^{p(i-1)})$.

If $F(\widetilde{\boldsymbol{\sigma}},\overline{\varepsilon}^{p(i-1)})<0$, elastic behavior assumption holds(loading elastically or unloading). So that $m=1.0$;

If $F(\widetilde{\boldsymbol{\sigma}},\overline{\varepsilon}^{p(i-1)})>0$ and $F(\boldsymbol{\sigma}^{(i-1)},\overline{\varepsilon}^{p(i-1)})=0$, it is plastic loading in this step. So that $m=0.0$;

If $F(\widetilde{\boldsymbol{\sigma}},\overline{\varepsilon}^{p(i-1)})>0$ and $F(\boldsymbol{\sigma}^{(i-1)},\overline{\varepsilon}^{p(i-1)})<0$, it converts from elastic loading to plastic loading in this step. So that m must be calculated from:

$$F(\boldsymbol{\sigma}^{(i-1)} + m \cdot \Delta\widetilde{\boldsymbol{\sigma}}, \overline{\varepsilon}^{p(i-1)}) = 0 \tag{39}$$

By using the Mises yield condition, m is defined as

$$\begin{cases} m = \frac{-a_1+\sqrt{a_1^2-4a_0\cdot a_2}}{2a_2} \\ a_0 = F(\boldsymbol{\sigma}^{(i-1)}, \overline{\varepsilon}^{p(i-1)}) \\ a_1 = (\mathbf{S}^{(i-1)})^T \cdot \Delta\widetilde{\mathbf{S}} \\ a_2 = \frac{1}{2}\Delta\widetilde{\mathbf{S}}^T \cdot \Delta\widetilde{\mathbf{S}} \end{cases} \tag{40}$$

in which, $\Delta\widetilde{\mathbf{S}}$ is the deviate stress of $\Delta\widetilde{\boldsymbol{\sigma}}$, $\mathbf{S}^{(i-1)}$ is the deviate stress of $\boldsymbol{\sigma}^{(i-1)}$.

(g.3) Claculate the elastic-plastic strain increment

$$\Delta\widehat{\boldsymbol{\varepsilon}} = (1-m) \cdot \Delta\boldsymbol{\varepsilon}^{(i)} \tag{41}$$

(g.4) Claculate the elastic stress increment

$$\Delta\boldsymbol{\sigma}^e = m \cdot \Delta\widetilde{\boldsymbol{\sigma}} \tag{42}$$

(g.5) Claculate the elastic-plastic stress increment

$$\Delta\boldsymbol{\sigma}^{ep} = \int_{\varepsilon^{(i-1)}}^{\varepsilon^{(i-1)}+\Delta\widehat{\varepsilon}} \mathbf{D} \cdot d\boldsymbol{\varepsilon} \approx \mathbf{D}(\boldsymbol{\sigma}^{(i-1)} + \Delta\boldsymbol{\sigma}^e, \overline{\varepsilon}^{p(i-1)}) \cdot \Delta\widehat{\boldsymbol{\varepsilon}} \tag{43}$$

(g.6) Claculate the effective plastic strain increment

$$\begin{cases} \Delta\overline{\varepsilon}^p = \frac{2}{3}\Delta\lambda \cdot \sigma_s \\ \Delta\lambda = \frac{(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \cdot \mathbf{D}^e \cdot \Delta\widehat{\boldsymbol{\varepsilon}}}{(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \cdot \mathbf{D}^e \cdot (\frac{\partial f}{\partial \boldsymbol{\sigma}}) + \frac{4}{9}\sigma_s^2 \cdot E^p} \end{cases} \tag{44}$$

(g.7) Claculate the stress and effective plastic strain of current step

$$\begin{cases} {}^{t+\Delta t}\boldsymbol{\sigma}^{(i)} = {}^{t+\Delta t}\boldsymbol{\sigma}^{(i-1)} + \Delta\boldsymbol{\sigma}^e + \Delta\boldsymbol{\sigma}^{ep} \\ {}^{t+\Delta t}\overline{\varepsilon}^{p(i)} = {}^{t+\Delta t}\overline{\varepsilon}^{p(i-1)} + \Delta\overline{\varepsilon}^p \end{cases} \tag{45}$$

(h) If the solution of this step touches the convergence condition, go to next loading step; else go to next iteration step of current loading step.

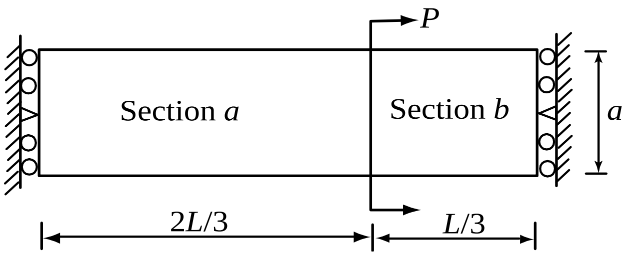

Figure 1: A rigidly supported bar

6 Numerical examples

6.1 A simple bar

A bar rigidly supported at two ends is subjected to an axial load as shown in figure 1, whose length is $L = 0.15m$, and width is $a = 0.01m$. The stress-strain relation in tension and compression is given in figure 2, where $E = 10^{10}N/m^2$, $\nu = 0.2$, $\sigma_{s0} = 2.0 \times 10^8 N/m^2$, $E^T = 10^9 N/m^2$. Assume that the displacement and strain are small, and the load is applied slowly. Figure 3 presents the numerical result obtained by the present method at the point of load application, which is in excellent agreement with the analytical solution. The analytical solution of this problem is:

$$\begin{cases} \sigma_a = \frac{u_p}{0.1}E & \sigma_b = \frac{u_p}{0.05}E & (\; if \;\; u_p < 0.001m) \\ \sigma_a = \frac{u_p}{0.1}E & \sigma_b = (\frac{u_p}{0.05} - 0.02)E_T + \sigma_{s0} & (\; if \;\; 0.001m < u_p < 0.002m) \\ \sigma_a = (\frac{u_p}{0.1} - 0.02)E_T + \sigma_{s0} & \sigma_b = (\frac{u_p}{0.05} - 0.02)E_T + \sigma_{s0} & (\; if \;\; u_p > 0.002m) \end{cases}$$

When the displacement u_p is less than $0.001m$, the whole plate is elastic; when u_p is between $0.001m$ and $0.002m$, section a is elastic while section b becomes plastic. And when u_p is greater than $0.002m$, the whole plate becomes plastic.

6.2 A Notched plate

A notched plate is subjected to the axial load as shown in Figure 4. The plate's length is $L = 0.04m$, width is $d = 0.02m$, and the notch's width is $a = 0.01m$, depth is $h = 0.0025m$. The material is assumed to be ideal-plastic with $E = 10^8 N/m^2$, $\nu = 0.2$ and $\sigma_{s0} = 3.0 \times 10^7 N/m^2$. Assuming that the displacement and strain are small, and the load is applied slowly. The numerical results show that plastic region initiates at notch, and then expands with the

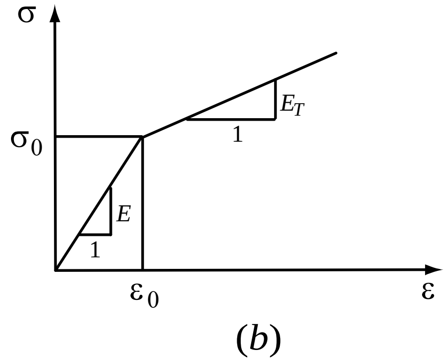

Figure 2: Stress-strain relationship

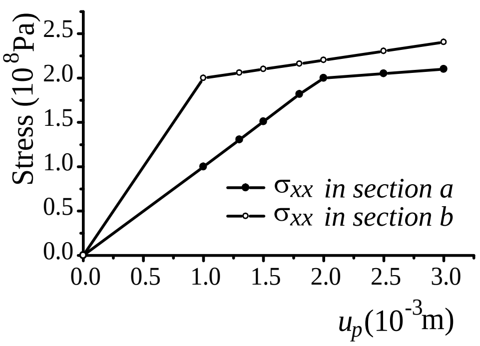

Figure 3: Stress-displacement relationship at the point of load application

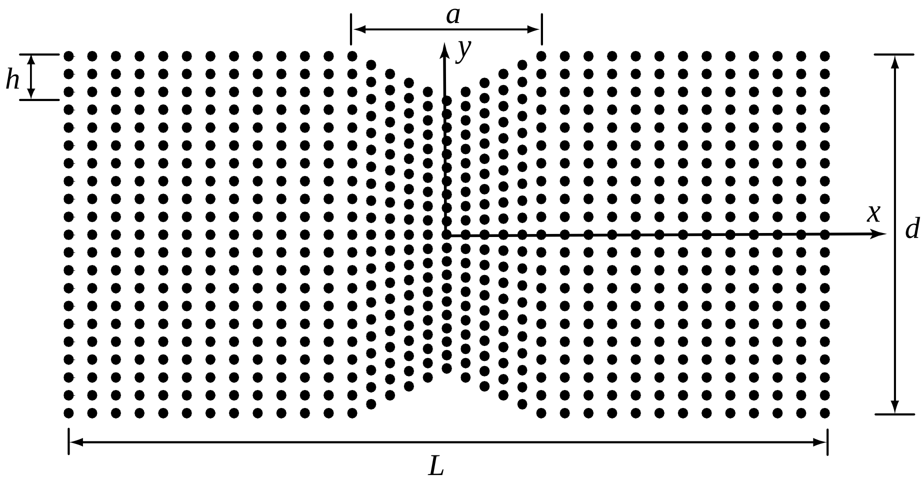

Figure 4: A notched plate

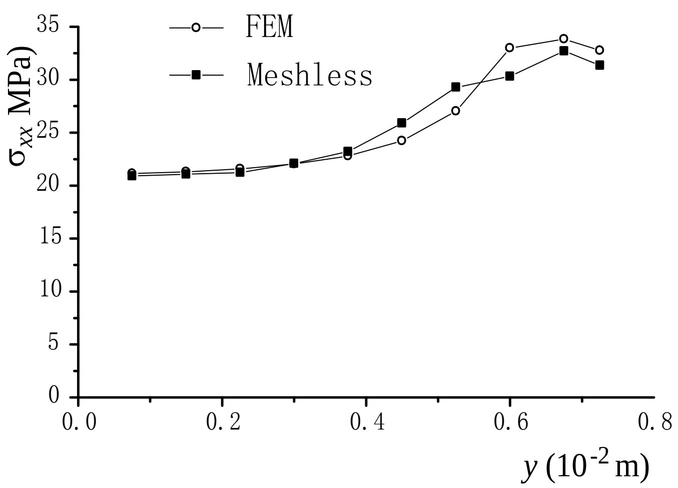

Figure 5: Stress σ_{xx} at $x = 0.0$

increase of the load. The limit load obtained is $2316N$, while the analytical solution is $2250N$. The numerical results obtained for $P = 1876N$ by the present method are compared with those obtained by FEM in Figures 5 and 6. Figure 7 illustrate the finite element mesh used in the FEM analysis. ALGOR is used in the FEM analysis.

7 Concluding remarks

The truly meshless method based on MLS approximate and direct collocation presented in this paper offers considerable promise for the solution of complicated materially-nonlinear problems in continuum mechanics. The present method is very computationally efficient, because only node is used to construct the discrete equations. Further work will be focused

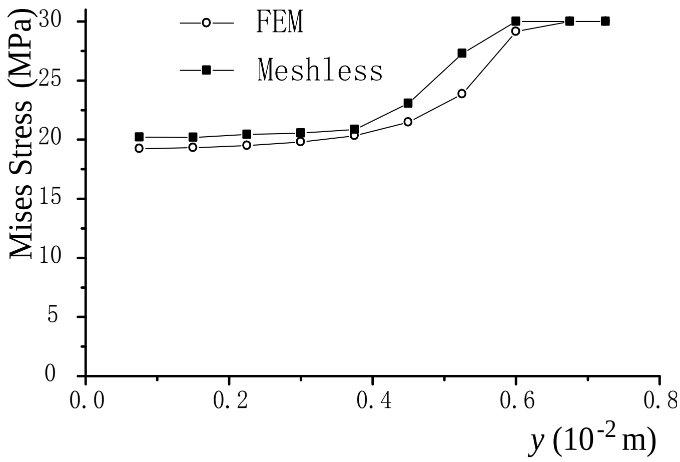

Figure 6: Mises-Stress at $x = 0.0$

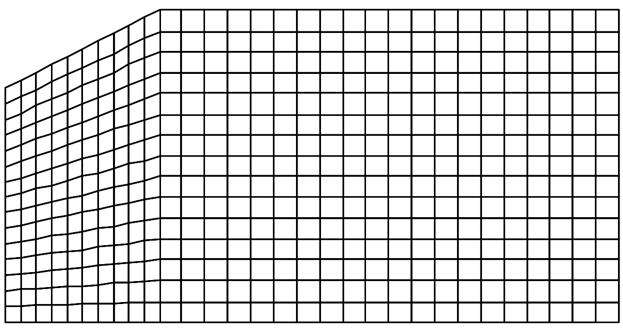

Figure 7: Finite element mesh

on the application of this method to the complex problems with both material and geometrical nonlinearity.

8 References

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