Slope stability analysis based on the rigid finite element method[†]

Xiong ZHANG
Associate professor
Department of Engineering Mechanics
Tsinghua University, Beijing 100084, P. R. China

Date paper written: June, 1997

Date paper revised: August, 1998

Correspondence to:

Xiong Zhang

Department of Engineering Mechanics

Tsinghua University

Beijing 100084, P. R. China

Telephone	+86-10-62784969 (H)
	+86-10-62782078 (O)
Fax	+86-10-62784969
Email	xzhang@tsinghua.edu.cn

[†] supported by National Natural Science Foundation of China under grant number 59509002

ABSTRACT

Limit equilibrium methods are widely used for the stability analysis of slopes, embankments, and excavations. These methods do not satisfy the overall equilibrium conditions, consequently, several assumptions regarding the interslice forces must be made for the problem to be solvable. Based on the ideas of the limit analysis and the rigid finite element method, a slope stability analysis method is proposed in this paper to avoid these drawbacks. A slope is treated as a number of slices with any arbitrary polyhedral shape connected by elastoplastic interfaces. All overall equilibrium conditions and yielding criteria are satisfied. Without imposing any assumptions regarding the interslice forces, the aforementioned shortcomings of limit equilibrium methods are avoided. Furthermore, the interslice forces are so adjusted by the limit analysis method that the mechanism of progressive failure of a slope, which has s significant effect on the overall factor of safety for brittle soil, is inherently considered in the present method. Nonlinear programming is also used to search for the true slip surface, which corresponds to the minimum factor of safety, among all possible slip surfaces. The formulation proposed here satisfies both the static and kinematic admissibility conditions without requiring any assumptions regarding the interslice forces. It can also be used to obtain the failure mechanism of a slope. The present method can be used to estimate the factor of safety of a complicated slope, the bearing capacity of foundations and the lateral earth pressure between a soil mass and adjoining retaining structure.

Keywords Bearing capacity, Earth pressure, Limit state design/analysis, Numerical modeling and analysis, slopes

1. Introduction

The determination of stability of slopes is a very important problem in rock and soil engineering. The methods of slices and the finite element method are the most frequently used methods for estimating the stability of slopes. Although the potential failure parts of a slope can be obtained by means of a finite element analysis the determination of a suitable measure and a set of rules from a finite element analysis to estimate the stability of slopes need to be further studied. Methods of slices are very simple and a quantitative index for stability, say factor of safety, can also be obtained, therefore they are very easily accepted by engineers.

In methods of slices, the slope is divided into N slices with $6N$-2 unknowns. The forces acting on a slice within a sliding soil mass are shown in figure 1, where

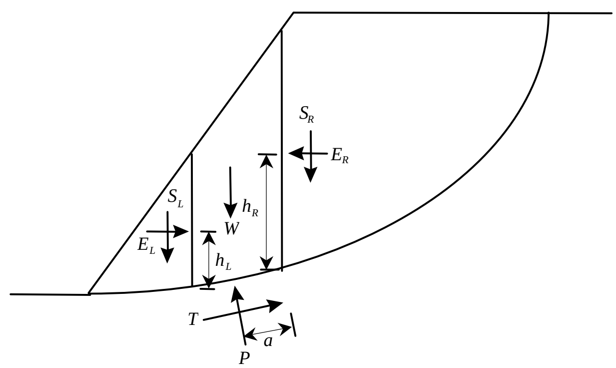

Figure 1: Forces acting on each slice

W = total weight of the slice.

P = total normal force acting on the base of the slice.

T = total shear force acting on the base of the slice.

a = offset distance from the normal force P to the right side of the slice.

E_L, E_R = horizontal interslice normal forces on the left and right sides of the slice, respectively.

S_L, S_R = vertical interslice shear forces on the left and right sides of the slice, respectively.

h_R, h_L = height of the interslice normal forces on the left and right sides of the slice, respectively.

This problem is indeterminate because only N Mohr-Coulomb failure criteria and $3N$ static equilibrium equations are available. Thus $2N$-2 assumptions must be made regarding the interslice forces for the problem to be solvable. Some of the conventional methods of slices neglect part of equilibrium conditions, and others make assumptions regarding the interslice forces. For example, the ordinary method of slices (Fellenius 1936) and Bishop's routine method (Bishop 1955) do not satisfy the condition of overall force equilibrium while the infinite slope analysis (Fredlund and Krahn 1977), the wedge analysis and Janbu's simplified method (Janbu 1954) do not satisfy the condition of overall moment equilibrium. Bishop's routine method (Bishop 1955) assumes that the interslice forces are horizontal, Janbu's method (Janbu 1973) assumes a line of thrust, and Morgenstern and Price (1965) assumes a functional relationship between the interslice shear forces and the interslice normal forces.

In recent years, Many efforts have been made to avoid the assumptions regarding the interslice forces (Ching 1992; Chuang 1992a, 1992b; Zhang 1994; Michalowski 1995). To avoid the shortcomings of limit equilibrium method, an alternative method for slope stability analysis is presented in this paper by combining the ideas of the rigid finite element method (Qian and Zhang 1991, 1995) and limit analysis (Zhang and Qian 1993). A slope is divided into N slices having any arbitrary polyhedral shape connected by elasto-plastic interfaces (Zhang 1992). This model satisfies all equilibrium conditions, the yielding criterion and the compatibility condition between the slices without requiring any assumptions regarding the interslice forces. The method is therefore theoretically more rigorous.

Rigid finite element method (RFEM), which is called the rigid bodies-springs model (RBSM) by Kawai (1978), came initially from the discrete element method (DEM) (Cundall, P. A. 1971). The DEM is a transient method, in which engineering problems are modeled as a large system of distinct interacting general shaped bodies. The dynamic contact topology of the bodies is determined by the solution of the equation of motion of every body, consequently, the DEM is a very computationally intensive procedure. Connecting the bodies by springs, as proposed in the RBSM of KAWAI (1978), significantly reduces the computational effort required. Qian and Zhang (1991) and Zhang and Lu (1996) established the mathematical basis of the method and classified it as a kind of finite element method where the rigid body displacement is taken as its displacement field, hence it is also called as rigid finite element method. Based on the ideas of RFEM, Zhang (1992) proposed a rigid body-elastoplastic interfaces model to simulate the mechanical behavior of the rock and soil structures, such as the bearing capacity, slope stability, retaining wall, etc. RFEM has been extended to simulate the nonlinear behavior of rock and soil structures (Zhang and Lu, 1998).

In the present method, limit analysis based on the lower bound theorem is used to estimate the factor of safety for a given slip surface while nonlinear programming is used to search the true slip surface corresponding to the minimum factor of safety among all possible surfaces. The formulation presented here satisfies both the static and kinematic admissibility of a discretized soil mass without requiring any assumptions. The method can easily treat more complicated problems, such as a slope with inhomogeneous properties, forces acting on the slope, bearing capacity of foundations and lateral earth pressure between a soil mass and adjoining retaining structures, etc.

2. Basic ideas of the rigid finite element method

Starting from the equilibrium equation of elastic problems, Qian and Zhang (1991, 1995) derived the formulations of the RFEM. Only a brief description of the basic ideas of the RFEM related to limit analysis is given in this section.

In the RFEM, a slope is discretized by a number of rigid slices with any arbitrary polyhedral shape connected by interfaces. Interslice surfaces and slip surfaces are all treated as interfaces. The displacement vector $\mathbf{u}(x,y)\ (=[u,v]^T)$ at any point $P(x,y)$ within a slice can be expressed by the translational displacement (u_g, v_g) at the centroid (x_g, y_g) of the slice and the rotational displacement θ of the slice as

$$\mathbf{u}(x,y) = \mathbf{N} \cdot \mathbf{u}_g \tag{1}$$

where $\mathbf{N} = \begin{bmatrix} 1 & 0 & y_g - y \\ 0 & 1 & x - x_g \end{bmatrix}$; $\mathbf{u}_g = [u_g, v_g, \theta]^T$

Figure 2 shows schematically the relative deformation of a pair of slices connected by an interface. Denoting the relative displacement at the point P in normal direction n between the top and bottom surfaces of the interface by δ_n and the relative displacement in tangential direction s by δ_s, the relative displacement vector $\Delta = [\delta_n, \delta_s]^{\mathrm{T}}$ can be expressed as

$$\Delta = \overline{\mathbf{U}}_j - \overline{\mathbf{U}}_i \tag{2}$$

where $\overline{\mathbf{U}}_i$ and $\overline{\mathbf{U}}_j$ are the displacement vector at point i and j in the local coordinate system n-s of the interface. Transforming the displacement vector at point i and j from the local coordinate system to the global coordinate system and substituting (1) into (2) gives

$$\Delta = \mathbf{B} \cdot \mathbf{X} \tag{3}$$

where $\mathbf{B} = \begin{bmatrix} -l_1 & -m_1 & -m_1(x-x_{g1})+l_1(y-y_{g1}) & l_1 & m_1 & m_1(x-x_{g2})-l_1(y-y_{g2}) \\ -l_2 & -m_2 & -m_2(x-x_{g1})+l_2(y-y_{g1}) & l_2 & m_2 & m_2(x-x_{g2})-l_2(y-y_{g2}) \end{bmatrix}$,

$\mathbf{X} = [u_{g1}, v_{g1}, \theta_1, u_{g2}, v_{g2}, \theta_2]^{\mathrm{T}}$,

l_1, m_1, l_2 and m_2 are the direction cosines in the *n* and s directions respectively.

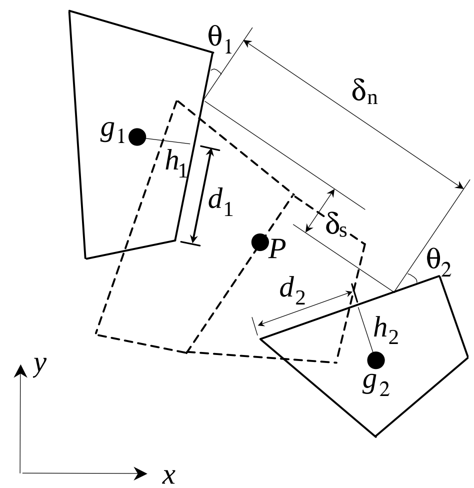

Figure 2: Relative displacement between slices

Equation (3) is the condition of compatibility between the slices. The normal stress on interfaces in the *s* direction can be neglected compared with other stress components, consequently, the stress vector **R** is given by

$$\mathbf{R} = [\sigma_n \ \ \tau_s]^{\mathrm{T}} \tag{4}$$

Based on the above equations, the variational principle for rigid finite element method (Zhang and Lu, 1996) leads to the overall equilibrium equation for elastic static problems. The same formulation also can be obtained directly from the equilibrium equation (Qian and Zhang 1991, 1995). Because slope failure is a progressive process, the stress obtained from elastic analysis can not be used directly to calculate the factor of safety. Instead, a limit analysis or an elasto-plastic analysis should be used.

3. Limit analysis

The conventional methods of slices assumes that the mobilized surfaces are all at the plastic limit state if their shear strength is reduced by the factor of safety *F*. Consequently, the normal and shear stresses are not independent on the mobilized surface, but related by the Mohr-Coulomb criteria. The mechanism of progressive failure of slopes is not included in the conventional methods of slices. In this paper limit analysis based on the lower theorem is used in conjunction with the rigid finite element method to obtain a plastic limit state.

The stability analysis of a slope should be a kind of limit analysis. When a slope is acted on by its weight and other applied forces, the failure of parts of the slope does not mean the failure of the complete slope, which can still bear the applied forces. The slope would completely lose its bearing capacity only when it became a mechanism, resulting from the propagation of the failure parts in the slope. This state can be obtained by elastoplastic or limit analysis.

From the limit analysis point of view, if the stress distributions in a slope satisfy the equilibrium conditions and do not violate the yielding criteria, all methods will give a statically admissible solution. According to the lower bound theorem of limit analysis, the limit load corresponding to a static admissible stress distribution should not be greater than the real one. Hence we should adjust the admissible stress distributions for a given slip surface to maximize the factor of safety, and then adjust the shape and location of the slip surface to find the true surface corresponding to the minimum factor of safety among all possible slip surfaces. This idea is the starting point of the present method.

A statically admissible stress distribution should satisfy the equilibrium conditions, consequently the stress distribution and the applied forces should satisfy the following virtual work equation (Zhang and Lu 1996) if they are in equilibrium

$$\sum_e \iiint_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV + \sum_j \iint_{\beta_j} \delta(\boldsymbol{\Delta}_j)^T \mathbf{R}_j ds - \lambda \sum_e \iiint_V \delta\mathbf{u}^T \mathbf{p}_e dV = 0 \tag{5}$$

where $\delta\boldsymbol{\varepsilon}$ represents the virtual strain vector in each slice, $\boldsymbol{\sigma}$ the stress vector in each slice, $\delta\mathbf{u}$ the virtual displacement vector, λ the load multiplier, $\mathbf{p}_e$ the reference volume load vector, $\mathbf{R}_j$ the stress vector on interface j, and $\delta(\mathbf{\Delta}_j)$ the virtual relative deformation at a point on interfaces. In (5), subscript e represents the slice e, and β_j the interface j. According to the rigid assumption for slices, the first item of the left-hand side of (5) equals zero. Substituting (3) and (1) into (5) leads to

$$-\sum_j \delta\mathbf{X}_j{}^T \iint_{\beta_j} \mathbf{B}_j{}^T \mathbf{R}_j ds + \lambda \sum_e \delta\mathbf{u}_g^T \iiint_V \mathbf{N}^T \mathbf{p}_e dV = 0 \tag{6}$$

where $\sum_j$ represents the assembly of all the interfaces, including all interslice surfaces and slip surfaces, and $\sum_e$ the assembly of all slices.

Substituting (3) and (4) into the first item of the left-hand side of (6) gives

$$\iint_{\beta_j} \mathbf{B}_j^T \mathbf{R}_j ds = \mathbf{C}_j^T \mathbf{V}_j \tag{7}$$

where

$$\mathbf{C}_j = \begin{bmatrix} -l_1 & -m_1 & d_1 & l_1 & m_1 & -d_2 \\ -l_2 & -m_2 & h_1 & l_2 & m_2 & -h_2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{V}_j = [E_j \quad S_j \quad M_j]^T$$

$$E_j = \iint_{\beta_j} \sigma_n ds, \quad S_j = \iint_{\beta_j} \tau_s ds, \quad M_j = \iint_{\beta_j} s\sigma_n ds$$

where E_j, S_j and M_j are the interslice normal force, interslice shear force and moment of the normal stress with respect to the left point of the interface β_j, respectively, see figure 3. Considering the arbitrariness of the virtual displacement and substituting (7) into (6) leads to

$$-\mathbf{C}^T \mathbf{V} + \lambda \mathbf{P} = 0 \tag{8}$$

where $\mathbf{C}$, $\mathbf{V}$ are matrices assembled by $\mathbf{C}_j$ and $\mathbf{V}_j$ for all interfaces, respectively, and $\mathbf{P} = \sum_e \iiint_V \mathbf{N}^T \mathbf{p}_e dV$.

Equation (8) is the overall force and moment equilibrium equation. By choosing the overall forces **V** as the basic variable to describe the interslice stress distribution, we do not impose any assumption regarding the location and inclination of interslice forces. Furthermore, the overall force equilibrium and moment equilibrium conditions are satisfied in the present method. After analyzing many slopes we find that the interslice moments have very little effect on the factor of safety in most cases, so that in these cases, only interslice forces are needed for calculating the factor of safety, and the matrix $\mathbf{C}_j$ and vector $\mathbf{V}_j$ simplify to

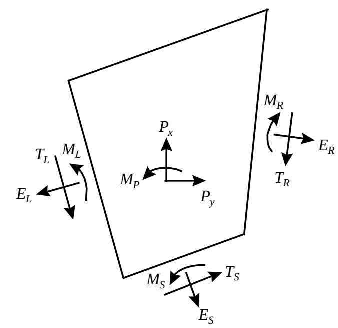

Figure 3: Forces acting on a slice in the present method

$$\mathbf{C}_j = \begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \\ -l_2 & -m_2 & l_2 & m_2 \end{bmatrix}, \quad \mathbf{V}_j = \begin{bmatrix} E_j \\ S_j \end{bmatrix} \tag{9}$$

Based on the above discussions, we should find a static admissible stress distribution **V** to maximize the load multiplier λ for a given slip surface, that is

$$\begin{aligned} & \mathit{Find} \;\; \lambda, \mathbf{V} = ? \\ & \max \;\; \lambda(\mathbf{V}) \\ & \mathit{subject\ to} \; \begin{cases} -\mathbf{C}^T\mathbf{V} + \lambda\mathbf{P} = 0 \\ \mathbf{M} \cdot \mathbf{V} \le \mathbf{K}_s \\ \lambda \ge 0 \end{cases} \end{aligned} \tag{10}$$

The modified Mohr-Coulomb criteria in integral form is shown in figure 4, and the matrix **M** and the vector $\mathbf{K}_s$ are given by

$$\mathbf{M} = diag(\mathbf{M}_j),$$

$$\mathbf{K}_s = [\mathbf{K}_1^T, \mathbf{K}_2^T, \cdots, \mathbf{K}_j^T, \cdots]^T,$$

$$\mathbf{M}_j = \begin{bmatrix} tg\varphi & 1 & 0 \\ tg\varphi & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K}_j = L_j \cdot [C, C, R_T]^T$$

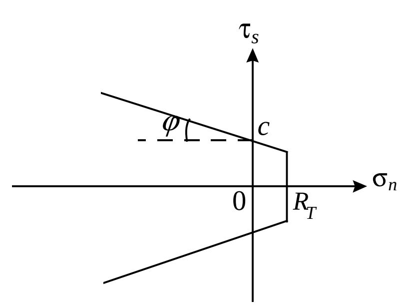

Figure 4: Modified Mohr-Coulomb condition

where L_j is the length of the interface j. Equation (10) represents a linear programming problem with free variables and can be solved directly by the improved simplex method proposed by Zhang and Zheng (1992). It automatically adjusts the statically admissible stress distribution **V** to obtain the maximum limit load multiplier for a given slip surface.

An alternative limit analysis method, the thermos-parameters method (Qian and Wang, 1990), can be used to reduce the total number of variables and constraints (Zhang and Qian 1993).

The dual linear programming of (10) can be expressed as

$$
\begin{aligned}
&\textit{Find}\ \ \dot{\mathbf{U}}, \dot{\beta} \\
&\min\ \ \lambda = \mathbf{K}_s^T \dot{\beta} \\
&\textit{subject to} \begin{cases} \mathbf{C}\dot{\mathbf{U}} - \mathbf{M}^T \dot{\beta} = 0 \\ \dot{\mathbf{U}}^T \mathbf{P} = 1 \\ \dot{\beta} \geq 0 \end{cases}
\end{aligned}
\tag{11}
$$

Where $\dot{\mathbf{U}}$ is the velocity of each slice when the slope is at a limit state. It defines the failure mechanism of the slope. Equation (11) also defines a linear programming problem with free variables which can be solved by the improved simplex method. The failure mechanism of a slope can be obtain from it.

For a stratified soil or if there is seepage, every slice can be sub-divided into sub-slices by the phreatic surfaces and the soil strata interfaces as shown in figure 5. The stress distribution on the sub-slice interfaces are also treated as free variables so that we can obtain the limit load multiplier for this complicated case. Figure 6 illustrates the complete subdivision of a slope.

A soil mass can be divided into slices, wedges, or arbitrary polyhedra, and the slip surface can also take any arbitrary shape in the present method. Consequently this method can be used to calculate the factor of safety of a complicated slope, the bearing capacity of foundations and the lateral earth pressure between a soil mass and adjoining retaining structure, etc.

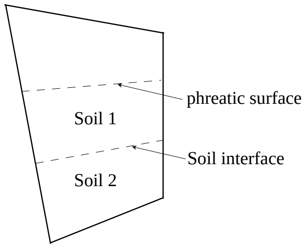

Figure 5: A slice is divided into several sub-slices for stratified soil or soil with seepage

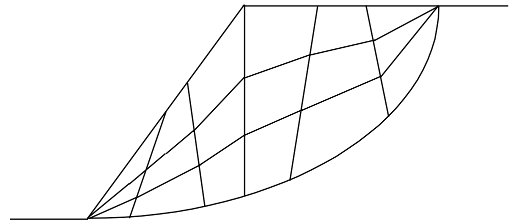

Figure 6: Every slice is divided into several sub-slices in vertical direction to obtain a more precise result

4. The factor of safety

The limit load multiplier λ defined here indicates that a slope will be at a limit state if it is acted on by load λ**P** (where **P** is a reference load), hence it is not the factor of safety *F* defined in the conventional methods of slices. The factor of safety *F* means that a slope will be in limit state if the shear strength of soil is reduced by *F* along the slip surface. For computational convenience, we introduce a reduction factor *F* into (10), that is

$$\mathbf{M}_j = \begin{bmatrix} tg\varphi/F & 1 & 0 \\ tg\varphi/F & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{K}_j = L_j \cdot [\frac{C}{F}, \frac{C}{F}, 0]^T \tag{12}$$

There are two ways to calculate the factor of safety *F*. Firstly, we can treat *F* as a variable and let λ equal 1 in (10), namely,

$$\begin{aligned} & Find \;\; F, \mathbf{V} = ? \\ & \max \;\; F(\mathbf{V}) \\ & subject\ to \;\; \begin{cases} -\mathbf{C}^T\mathbf{V} + \mathbf{P} = 0 \\ \mathbf{MV} \le \mathbf{K}_s \\ F \ge 0 \end{cases} \end{aligned} \tag{13}$$

where the sub-matrices of **M** and $\mathbf{K}_s$ are given in (12). Equation (13) defines a nonlinear programming problem and significant computational effort is required to obtain its solution. Secondly, to avoid solving the nonlinear programming problem, an initial value of *F* is chosen, and then the linear programming problem (10) is solved. The factor of safety will be obtained if λ solved from (10) is equal to 1. Otherwise modify the value of *F* and solve (10) again until λ is equal to 1. After a few iterations, the factor of safety F is obtained with the value of λ is equal to 1.

5. Numerical examples

Two numerical examples are given in here to investigate the accuracy and the scope of applicability of the present method.

1) Taylor (1948) analyzed the stability of dry homogeneous material for the slope shown in figure 7 by using the ordinary method of slices, the friction circle and log spiral methods of analysis. Chen (1975) compared the results of each limit equilibrium analysis with an upper bound analysis. The comparisons show that the results of the limit equilibrium method are very close to the upper bound solutions. The same slope is also analyzed by using the present method with a circle slip surface passed through toe and the results are compared with those obtained by the limit equilibrium methods and upper bound analysis in Table 1. It can be seen that the present method gives lower factors of safety for the slopes with slope angle β > 75°, but gives higher solutions if the slope angle β is smaller than 60°. This means that the slip surface will pass below the toe if the slope angle is smaller than 60° in the present method.

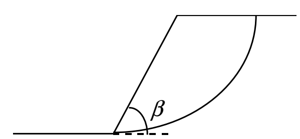

Figure 7: A homogeneous slope

Table 1: Comparison of Stability Number N = $\gamma H/C$

Slope angle β(°)	Friction angle ϕ(°)	Limit equilibrium			Limit analysis	This paper
		Slices	ϕ circle	Logspiral	Logspiral	Circle
90	0	3.83	3.83	3.83	3.83	3.49
	5	4.19	4.19	4.19	4.19	3.83
	15	5.02	5.02	5.02	5.02	4.62
	25	6.06	6.06	6.06	6.06	5.60
75	0	4.57	4.57	4.57	4.56	4.41
	5	5.13	5.13		5.14	4.99
	15	6.49	6.52		6.57	6.46
	25	8.48	8.54		8.58	8.55
60	0	5.24	5.24	5.24	5.25	5.58
	5	6.06	6.18	6.18	6.16	6.56
	15	8.33	8.63	8.63	8.63	9.23
	25	12.20	12.65	12.82	12.74	13.68
45	0	5.88	5.88*	5.88*	5.53*	6.77
	5	7.09	7.36		7.35	8.35
	15	11.77	12.04		12.05	13.37
	25	20.83	22.73		22.90	24.74

* Critical failure surface passes below toe.

2) Figure 8 shows the section of a strip footing on a homogeneous, weightless, purely cohesive soil, together with the failure mechanism proposed by Drucker (1953). Six triangular slices (rigid finite elements) are used to to calculate the bearing capacity *p*, see figure 8. Table 2 compares the bearing capacity obtained by the present method with Prandtl's value and Chuang's value, and they are in excellent agreement. Figure 9 shows the interslice forces of every slice obtained from (10), and figure 10 shows the velocity of every slice obtained from (11), which gives the failure mechanism of the strip footing. In this study, the interslice moments have no effect on the value of the bearing capacity *p*..

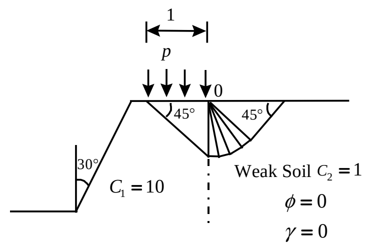

Figure 8: A strip footing

Table 2: Bearing capacity of the strip footing

Prandtl's value	Chuang's value	Present method
23.5708	23.5760	23.5759

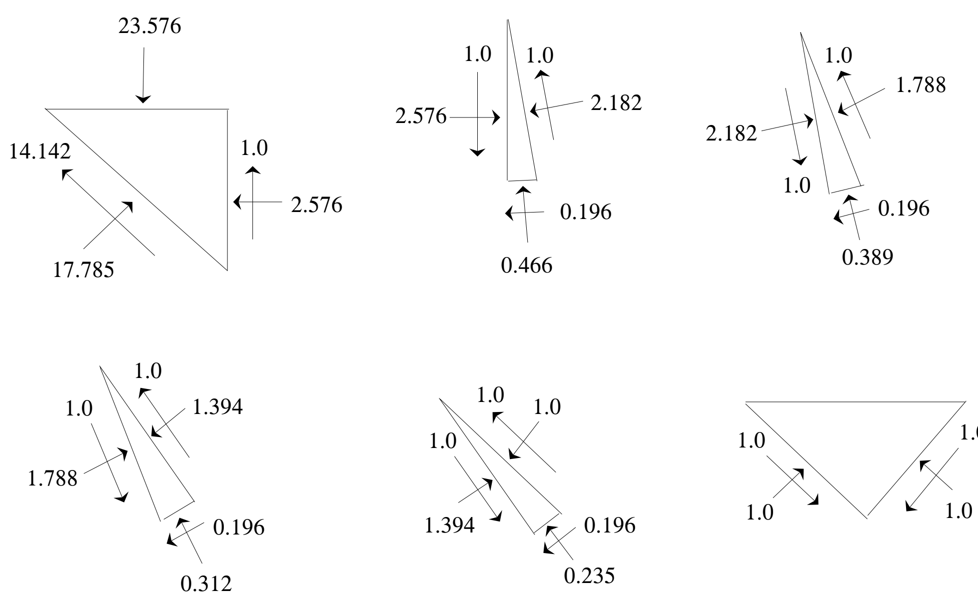

Figure 9: Inter-slice forces

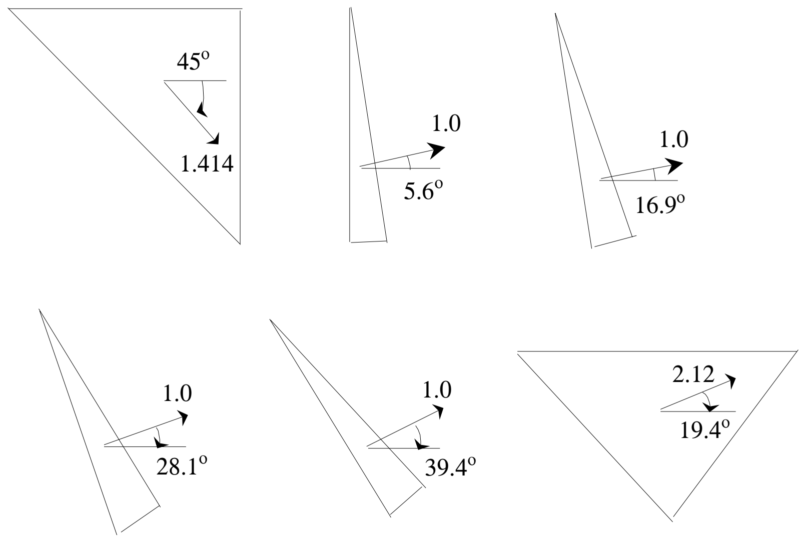

Figure 10: Velocity of every slice

7. Concluding Remarks

All conventional methods of slices have to make assumptions regarding the interslice forces to solve a slope stability problem, hence experience is required to make reasonable assumptions regarding the locations and inclinations of the interslice forces. In the present method, the overall interslice forces and moments are used to describe the interslice stress distribution and they are adjusted by linear programming to satisfy overall force and moment equilibrium as well as the Mohr-Coulomb criterion and no assumptions are imposed on the interslice forces. Furthermore, the progressive failure mechanism is inherently taken into account via limit analysis and the solution obtained is both kinematically and statically admissible. In this way, the present method is theoretically more rigorous and simple. The method can be used in slope stability analysis as well as in bearing capacity and earth pressure analysis.

References

Bishop, A. W. (1955). The use of slip circle in the stability analysis of slopes. *Geotechnique*, 5, No.1, 7-17.

Chen, W. F. (1975). Limit analysis and soil plasticity. Amsterdam: Elsevier.

Ching, S. C. (1992). Discrete element method for slope stability analysis. *Journal of Geotechnical engineering*, 118, No.12, 1889-1905

Chuang, P. H. (1992a). Stability analysis in geomechanics by linear programming. I: Formulation, *Journal of Geotechnical Engineering*, 118, No.11, 1696-1715.

Chuang, P. H. (1992b). Stability analysis in geomechanics by linear programming. II: Application, *Journal of Geotechnical Engineering*, 118, No.11, 1716-1726.

Craig, R. F. (1983). Soil Mechanics, 3rd edn. Van Nostrand Reinhold (UK).

Cundall, P. A. (1971). A computer model for simulating progressive, large-scale movements in block rock system, Proc. Int. Symp. On Rock Fracture, Nancy, France, 1, 8-17.

Drucker, D. C. (1953). Limit analysis of two and three dimensional soil mechanics problems. *J. Mech. Physics of Solids*, 1, 217-227.

Fellenius, W. (1936). Calculation of the stability of earth dams, Proc. Second Congress on Large Dams, 4, 445-463.

Fredlund, D. G. and Krahn, J. (1977). Comparison of slope stability methods of analysis, *Can. Geotech. J.*, 14, No.3, 429-439.

Janbu, N. (1954). Application of composite slip surface for stability analysis. European Conf. on Stability of Earth Slopes, 3, 43-49.

Janbu, N. (1973). Slope stability computations. In (Eds. Hirschfeld R.C. and Poulos S.J.) Embankment Dam Engineering, Casagrande Volume, Wiley, 47-86.

Kawai, T. (1978). New discrete models and their application to seismic response analysis of structures, *Nuclear engineering and design*, 48, 207-229

Mangasarian, O. L. (1969). Nonlinear programming, McGraw-Hill, New York, NY.

Michalowski, R. L. (1995). Slope stability analysis: a kinematical approach. *Geotechnique*, 45, No.2, 283-293.

Morgenstern, N. R. and Price, V. E. (1965). The analysis of the stability of general slip surfaces. *Geotechnique*, 15, No.1, 79-93.

Nash, D. (1987). A comparative review of limit equilibrium methods of stability analysis. In (Eds. Anderson, M. G. & Richards, K. S.) Slope Stability. John Wiley & Sons Ltd.

Qian L.X. and Wang Z.B. (1990). Structural limit analysis and shakedown analysis: a thermo-parameters method. Proceedings of 1990 Pressure Vessels and Piping Conference.

PVP’90, Nashnille, USA.

Qian L.X. and Zhang X. (1991). Rigid finite element method in structural analysis, Proc. Of the 2nd Asia Pacific Conference on Computational Mechanics, Hong Kong, 1165-1171

Qian L.X., Zhang X. (1995). Rigid finite element method and it’s applications in engineering. *Acta Mechanica Sinica*, 11, No.1, 44-50.

Taylor, D. W. (1948). Fundamentals of Soil Mechanics. Wiley, New York.

Zhang X. (1992). Limit analysis method in stability of rock slopes: rigid bodies-elastoplastic seams model. *Rock and Soil Mechanics* (in Chinese), 13, No.1, 66-73.

Zhang X. (1994). Improved Slice Method for Slope Stability Analysis, *Chinese Journal of Geotechnical Engineering*, 16, No.3, 84-92.

Zhang X. and Lu M.W. (1996). Variational principles for rigid finite element. Proceedings of Third Asian-Pacific Conference on Computational Mechanics, Seoul, Korea, 61-66.

Zhang X. and Lu M.W. (1998). Block-interface model for nonlinear numerical simulations of rock structures, *International Journal of Rock Mechanics and Mining Sciences,* to be present.

Zhang X., Qian L.X. (1993). Rigid finite element and limit analysis. *Acta Mechanica Sinica*, 9, No.2, 156-162.

Zhang X., Zheng L.S. (1992). An algorithm for solving linear programming with free variables. Proceedings of International Conference on Education, Practice and Promotion of Computational Methods in Engineering Using Small Computers, Dalian, China, 164-169.

List of notations

N - number of slices

W - total weight of a slice

P - total normal force acting on the base of the slice.

T - total shear force acting on the base of the slice.

a - offset distance from the normal force P to the right side of the slice.

E_L, E_R - horizontal interslice normal forces on the left and right sides of the slice, respectively.

S_L, S_R - vertical interslice shear forces on the left and right sides of the slice, respectively.

h_R, h_L - height of the interslice normal forces on the left and right sides of the slice, respectively.

u, v - the translational displacements at any point $P(x, y)$ within the slice

u_g, v_g - the translational displacements at the centroid (x_g, y_g) of the slice

θ - the rotational displacements of the slice

$\overline{\mathbf{U}}_i$ - the displacement vector at point i in local coordinate system n-s of the interface

Δ - the relative displacement vector between the top and bottom surfaces of the interface

l_1, m_1 - the direction cosines of n direction

l_2, m_2 - the direction cosines of s direction

σ_n, τ_s - the normal stress and the tangential stress on the interface, respectively.

$\delta\varepsilon$ - the virtual strain vector of the slice

σ - the stress vector of the slice

$\delta\mathbf{u}$ - the virtual displacement vector

$\mathbf{p}_e$ - the reference volume load vector

λ - the load multiplier

$\delta(\Delta_j)$ - the virtual relative deformation at a point on the interface j

$\mathbf{R}_j$ - the stress vector of the interface j

E_j - the interslice normal force

S_j - the interslice shear force

M_j - the moment of the normal stress with respect to the left point of the interface j

L_j - the length of the interface j

C, tgϕ - shear strength of the soil

R_T – Tensile strength of the soil

F - factor of safety

$\dot{\mathbf{U}}$ - velocity of each slice