无网格法

典型无网格法

无网格法总结

	伽辽金法	配点法	局部彼得罗夫-伽辽金法	加权最小二乘法	边界积分方程
MLS	DEM[47]; EFG[48]	FPM[94, 96]; DAM[118]; LSCM[91]; MPCM[119, 120]; Kim[121]	MLPG[9, 103, 104]; PG DEM[122]	WLSM[111, 112, 123]; LSMM[110, 113]	BNM[114, 115]; BCM[124]; LBIE[107, 108]; HBNM[117]
核近似		SPH[92, 93]			
重构核近似	RKPM[11, 49, 73]	FCM[125, 126]; HCM[127, 128]			
单位分解	PUM[15]; GFEM[18, 129]; XFEM[22, 23, 25]	hp-Meshless Cloud[97]	Finite spheres[130, 131]		
径向基函数	Wendland[132]	Kansa[133]; Franke[134]; Zhang[33]			BPM[116]
点插值	PIM[34]; RPIM[32]	RPCM[135]; PICM[136]	LRPIM[137]	Tan[138]	BRPIM[139]
Sibson插值	NEM[36, 39]		PGNE[140]		
非Sibson插值	NNGM[51]		NNPG[141]		
Kriging插值			Lam[43]; Gu[44]		
NURBS	Shaw[46]				

典型无网格法

- 伽辽金型无网格法
- 配点型无网格法
- 基于局部弱形式和边界积分方程的无网格法
- 最小二乘无网格法
- 物质点法

伽辽金型无网格法

- EFG
  - MLS
  - Galerkin
- RKPM
  - RK
  - Galerkin
- PIM
  - PI
  - Galerkin

伽辽金型无网格法

等效积分弱形式（虚位移原理）

$$\delta\Pi(\boldsymbol{u}) = \int_{\Omega} (\delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} - \delta\boldsymbol{u}^{\mathrm{T}}\bar{\boldsymbol{f}})\mathrm{d}\Omega - \int_{\Gamma_t} \delta\boldsymbol{u}^{\mathrm{T}}\bar{\boldsymbol{t}}\mathrm{d}\Gamma = 0$$

MLS近似：$u(\boldsymbol{x}) = \boldsymbol{N}(\boldsymbol{x})\boldsymbol{d}$

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{P}$$

$$\boldsymbol{K} = \int_{\Omega} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}\mathrm{d}\Omega$$

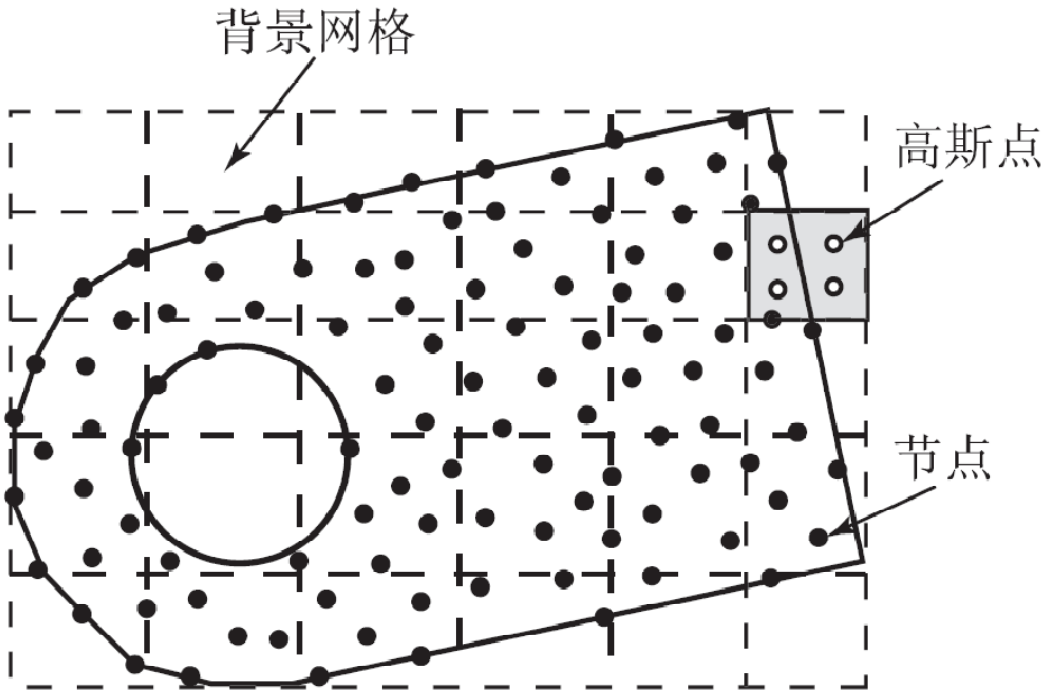
$$\boldsymbol{P} = \int_{\Omega} \boldsymbol{N}^{\mathrm{T}}\bar{\boldsymbol{f}}\mathrm{d}\Omega + \int_{\Gamma_t} \boldsymbol{N}^{\mathrm{T}}\bar{\boldsymbol{t}}\mathrm{d}\Gamma$$

- 计算量大
- 精度高，稳定性好
- 需要背景网格进行积分
- 系数矩阵对称
- 不易施加本质边界条件处理

伽辽金型无网格法 — 数值积分

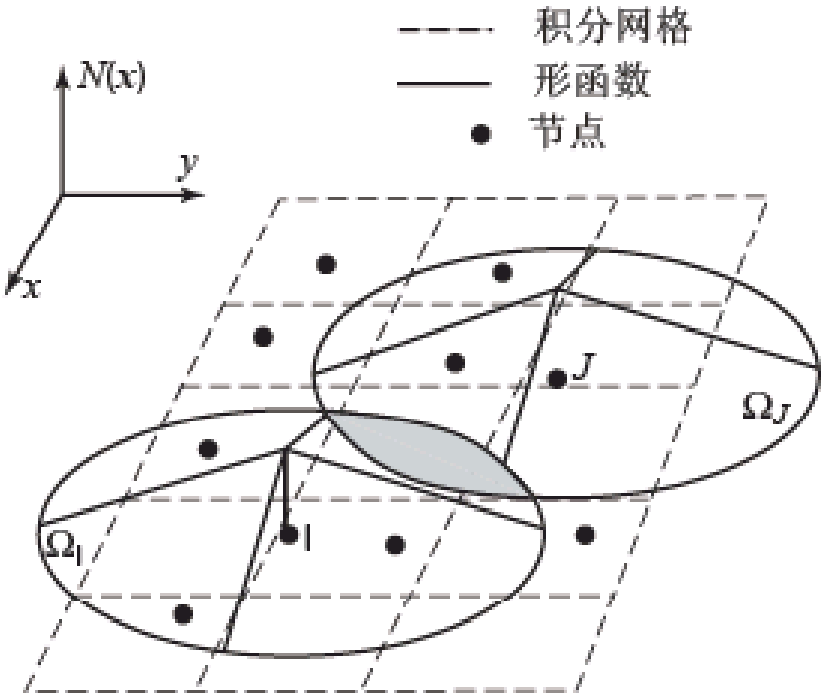
$$\int_{\Omega} \boldsymbol{f}(\boldsymbol{x})\mathrm{d}\Omega = \sum_{l=1}^{n_q} \boldsymbol{f}(\boldsymbol{x}_{\xi_l}) w_l$$

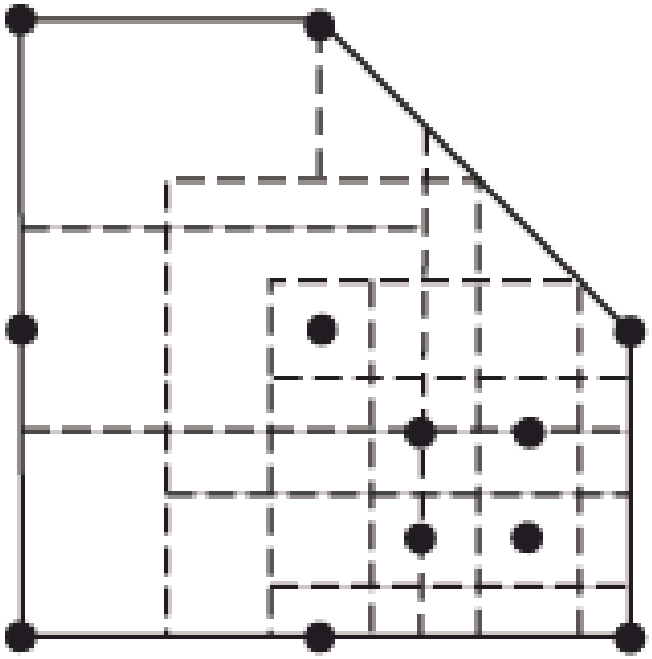
- 背景网格积分



伽辽金型无网格法 — 数值积分

$$\int_{\Omega} N_{I,i} N_{J,j} \mathrm{d}\Omega$$



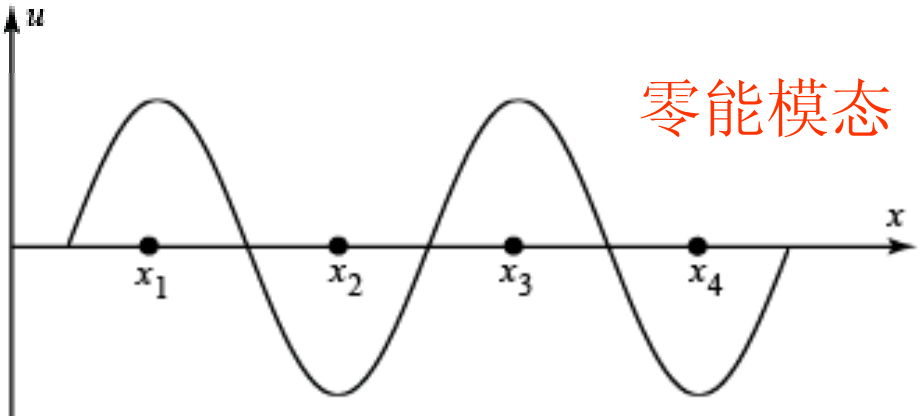


伽辽金型无网格法 — 数值积分

- 节点积分

$$\boldsymbol{K}=\sum_{I=1}^{N}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x}_I)\boldsymbol{D}\boldsymbol{B}(\boldsymbol{x}_I)\Delta\Omega_I$$

$$\boldsymbol{P}=\sum_{I=1}^{N}\boldsymbol{N}^{\mathrm{T}}(\boldsymbol{x}_I)\bar{\boldsymbol{f}}(\boldsymbol{x}_I)\Delta\Omega_I+\sum_{I=1}^{N_t}\boldsymbol{N}^{\mathrm{T}}(\boldsymbol{x}_I)\bar{\boldsymbol{t}}(\boldsymbol{x}_I)\Delta\Gamma_I$$



零能模态

稳定化方案

$$\delta\Pi_s(u_i)=\delta\Pi(u_i)+\frac{2\alpha_s l_c^2}{E}\int_\Omega\delta\sigma_{ij,j}(\sigma_{ij,j}+\bar{f}_i)\mathrm{d}\Omega=0$$

伽辽金型无网格法 — 数值积分

光滑应变稳定化方案

将节点$\boldsymbol{x}_L$处的应变取为节点$\boldsymbol{x}_L$的邻域$\boldsymbol{\Omega}_L$内应变的加权平均

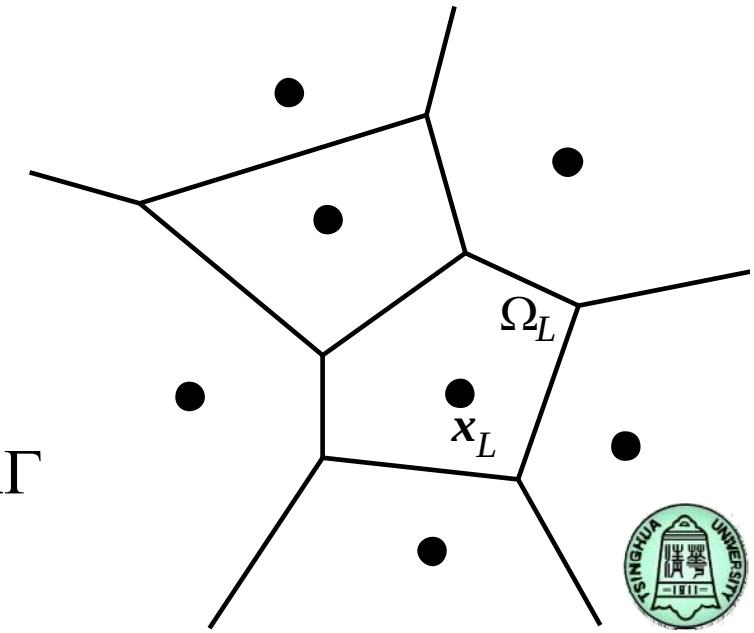
$$\tilde{\varepsilon}_{ij}=\int_{\Omega}\varepsilon_{ij}\phi(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x}_L)\mathrm{d}\Omega \qquad \phi(\boldsymbol{x};\ \boldsymbol{x}-\boldsymbol{x}_L)=\begin{cases}1/\Delta\Omega_L & \boldsymbol{x}\in\Omega_L\\ 0 & \boldsymbol{x}\notin\Omega_L\end{cases}$$

$$=\frac{1}{2\Delta\Omega_L}\int_{\Omega_L}\left\{\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right\}\mathrm{d}\Omega=\frac{1}{2\Delta\Omega_L}\int_{\Gamma_L}(u_i n_j+u_j n_i)\mathrm{d}\Gamma$$

$$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}_L)=\sum_{I\in G_L}\widetilde{\boldsymbol{B}}_I(\boldsymbol{x}_L)\boldsymbol{u}_I$$

$$\widetilde{\boldsymbol{B}}_I(\boldsymbol{x}_L)=\begin{bmatrix}\tilde{b}_{I1}(\boldsymbol{x}_L) & 0\\ 0 & \tilde{b}_{I2}(\boldsymbol{x}_L)\\ \tilde{b}_{I2}(\boldsymbol{x}_L) & \tilde{b}_{I1}(\boldsymbol{x}_L)\end{bmatrix}$$

$$\tilde{b}_{Ii}(\boldsymbol{x}_L)=\frac{1}{\Delta\Omega_L}\int_{\Gamma_L}N_I(\boldsymbol{x})n_i(\boldsymbol{x})\mathrm{d}\Gamma$$



伽辽金型无网格法 — 数值积分

- 单位分解积分

1. 函数$\psi_k(\boldsymbol{x})$只定义在子域Ω_k上；
2. 子域Ω_k相互重叠，且它们完全覆盖了域Ω;
3. 函数$\psi_k(\boldsymbol{x})$满足单位分解条件

$$\int_{\Omega} f(\boldsymbol{x})\mathrm{d}\Omega = \sum_{k=1}^{l}\int_{\Omega\cap\Omega_k} \psi_k(\boldsymbol{x})\, f(\boldsymbol{x})\mathrm{d}\Omega$$

伽辽金型无网格法 — 位移边界条件的处理

$$N_I(\boldsymbol{x}_J) \neq \delta_{IJ} \qquad u^h(\boldsymbol{x}_I) \neq u_I$$

- 拉格朗日乘子法

$$\Pi_\lambda(\boldsymbol{u},\lambda) = \Pi(\boldsymbol{u}) + \int_{\Gamma_u} \lambda^{\mathrm{T}}(\boldsymbol{u}-\bar{\boldsymbol{u}})\mathrm{d}\Gamma$$

$$\delta\Pi_\lambda(\boldsymbol{u},\lambda) = \delta\Pi(\boldsymbol{u}) + \int_{\Gamma_u} \delta\lambda^{\mathrm{T}}(\boldsymbol{u}-\bar{\boldsymbol{u}})\mathrm{d}\Gamma + \int_{\Gamma_u} \delta\boldsymbol{u}^{\mathrm{T}}\lambda\mathrm{d}\Gamma = 0$$

$$\lambda(\boldsymbol{x}) = \boldsymbol{N}^\lambda(s)\Lambda \quad \delta\lambda(\boldsymbol{x}) = \boldsymbol{N}^\lambda(s)\delta\Lambda$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G}^{\mathrm{T}} \\ \boldsymbol{G} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{d} \\ \boldsymbol{\Lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{Bmatrix}$$

$$\boldsymbol{G} = \int_{\Gamma_u} \boldsymbol{N}_\lambda^{\mathrm{T}}\boldsymbol{N}\mathrm{d}\Gamma, \quad \boldsymbol{Q} = \int_{\Gamma_u} \boldsymbol{N}_\lambda^{\mathrm{T}}\bar{\boldsymbol{u}}\mathrm{d}\Gamma$$

伽辽金型无网格法 — 位移边界条件的处理

- 修正变分原理

$$\delta\Pi_\lambda(\boldsymbol{u},\lambda)=\delta\Pi(\boldsymbol{u})+\int_{\Gamma_u}\delta\lambda^{\mathrm{T}}(\boldsymbol{u}-\bar{\boldsymbol{u}})\mathrm{d}\Gamma+\int_{\Gamma_u}\delta\boldsymbol{u}^{\mathrm{T}}\lambda\mathrm{d}\Gamma=0$$

$$\lambda=-\boldsymbol{t}\quad \text{在}\Gamma_u\text{上}$$

$$=\delta\Pi(\boldsymbol{u})-\int_{\Gamma_u}\delta\boldsymbol{t}^{\mathrm{T}}(\boldsymbol{u}-\bar{\boldsymbol{u}})\mathrm{d}\Gamma+\int_{\Gamma_u}\delta\boldsymbol{u}^{\mathrm{T}}\boldsymbol{t}\mathrm{d}\Gamma=0$$

$$(\boldsymbol{K}+\boldsymbol{K}_t)\boldsymbol{d}=\boldsymbol{P}+\boldsymbol{P}_t$$

$$\boldsymbol{K}_t=-\int_{\Gamma_u}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{n}\boldsymbol{D}\boldsymbol{B}\mathrm{d}\Gamma-\int_{\Gamma_u}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{n}^{\mathrm{T}}\boldsymbol{N}\mathrm{d}\Gamma$$

$$\boldsymbol{P}_t=-\int_{\Gamma_u}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{n}^{\mathrm{T}}\bar{\boldsymbol{u}}\mathrm{d}\Gamma$$

$$\boldsymbol{n}=\begin{bmatrix}n_x & 0 & n_y\\ 0 & n_y & n_x\end{bmatrix}$$

伽辽金型无网格法 — 位移边界条件的处理

- 罚函数法

$$\delta\Pi_p(\boldsymbol{u}) = \delta\Pi(\boldsymbol{u}) + \alpha\int_{\Gamma_u}\delta\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{u}-\bar{\boldsymbol{u}})\mathrm{d}\Gamma = 0$$

$$(\boldsymbol{K}+\boldsymbol{K}_p)\boldsymbol{d} = \boldsymbol{P}+\boldsymbol{P}_p$$

$$\boldsymbol{K}_p = \alpha\int_{\Gamma_u}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{N}\mathrm{d}\Gamma$$

$$\boldsymbol{P}_p = \alpha\int_{\Gamma_u}\boldsymbol{N}^{\mathrm{T}}\bar{\boldsymbol{u}}\mathrm{d}\Gamma$$

配点型无网格法

- FPM
  - MLS
  - Collocation
- SPH
  - KA
  - Collocation
- Hp meshless clouds
  - PU
  - Collocation

配点型无网格法 — FPM

微分方程在域内节点处满足，边界条件在边界节点处满足

$$\sigma_{ij,j}(\boldsymbol{x}_k)+\bar{f}_i(\boldsymbol{x}_k)=0 \quad \boldsymbol{x}_k\in\Omega, i,j=1,2,3; k=1,2,\cdots,N_\Omega$$

$$\sigma_{ij}(\boldsymbol{x}_k)n_j=\bar{t}_i(\boldsymbol{x}_k) \quad \boldsymbol{x}_k\in\Gamma_t, i,j=1,2,3; k=1,2,\cdots,N_t$$

$$u_i(\boldsymbol{x}_k)=\bar{u}_i(\boldsymbol{x}_k) \quad \boldsymbol{x}_k\in\Gamma_u, i=1,2,3; k=1,2,\cdots,N_u$$

- 只需要在各节点处计算形函数及其导数，不需要任何积分，易于实现且计算效率高；
- 位移边界条件很容易实现，不需要特殊处理；
- 需要在各节点处计算形函数的二阶导数；
- 系数矩阵是非对称的；
- 在某些情况下，方程是病态的，其解不稳定，且精度较差。

配点型无网格法 — FPM

稳定化

$$\sigma_{ij,j}(\boldsymbol{x}_k)+\bar{f}_i(\boldsymbol{x}_k)-\frac{1}{2}h_l\frac{\partial}{\partial x_l}\left[\sigma_{ij,j}(\boldsymbol{x}_k)+\bar{f}_i(\boldsymbol{x}_k)\right]=0$$

$$\sigma_{ij}(\boldsymbol{x}_k)n_j-\frac{1}{2}h_l n_l\left[\sigma_{ij,j}(\boldsymbol{x}_k)+\bar{f}_i(\boldsymbol{x}_k)\right]=\bar{t}_i(\boldsymbol{x}_k)$$

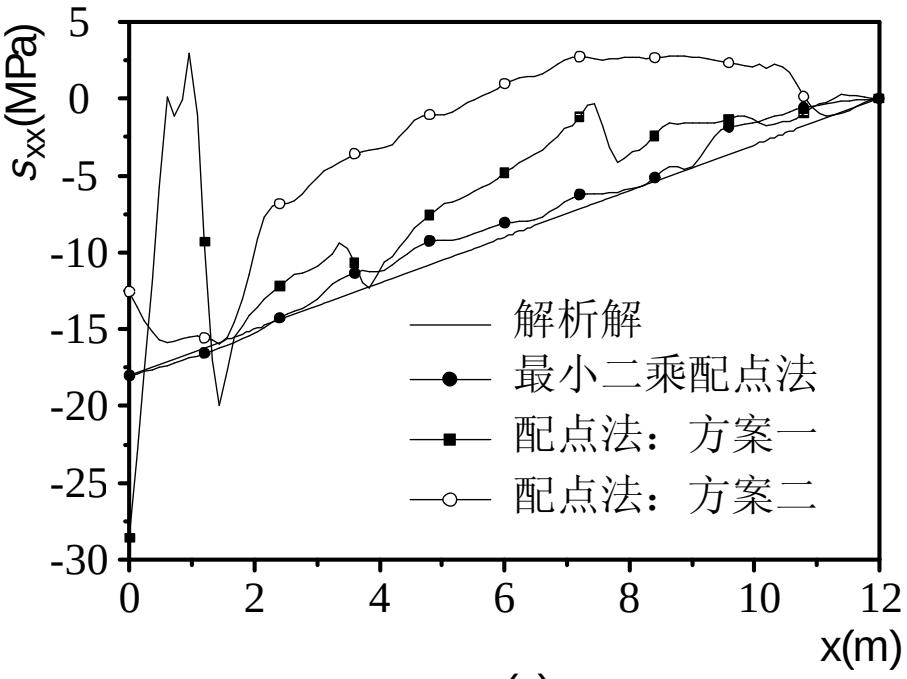
$$u_i(\boldsymbol{x}_k)=\bar{u}_i(\boldsymbol{x}_k)$$

配点型无网格法 — 最小二乘配点

$$\sigma_{ij,j}(\boldsymbol{x}_k)+\bar{f}_i(\boldsymbol{x}_k)=0 \qquad \boldsymbol{x}_k\in\Omega,\ i,j=1,2;\ k=1,2,\cdots,N+N_a$$

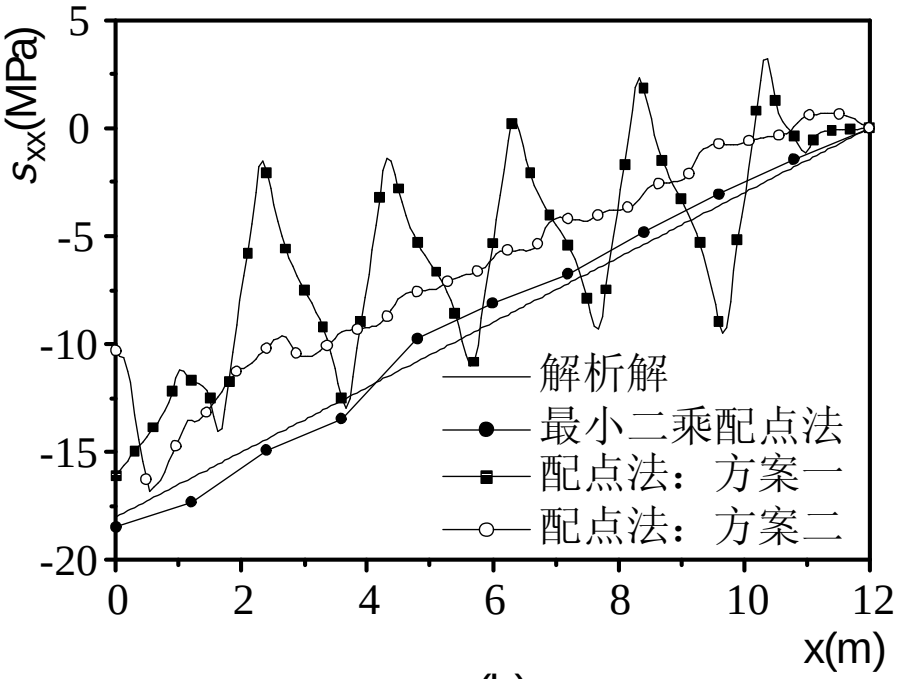
$$\sigma_{ij}(\boldsymbol{x}_k)n_j=\bar{t}_i(\boldsymbol{x}_k) \qquad \boldsymbol{x}_k\in\Gamma_t,\ i,j=1,2;\ k=1,2,\cdots,N_t$$

$$u_i(\boldsymbol{x}_k)=\bar{u}_i(\boldsymbol{x}_k) \qquad \boldsymbol{x}_k\in\Gamma_u,\ i=1,2;\ k=1,2,\cdots,N_u$$



(a)

$N = 48,\ N_a = 21$



(b)

$N = 107,\ N_a = 64$

配点型无网格法 — SPH

无体力时的运动方程

$$\nabla \cdot \boldsymbol{\sigma} = \rho \dot{\boldsymbol{v}}$$

配点法

$$\left(\nabla \cdot \boldsymbol{\sigma}\right)_I = \rho_I \dot{\boldsymbol{v}}_I \quad \Leftarrow \quad \nabla \cdot \boldsymbol{\sigma} \approx \sum_{J=1}^{N} \frac{m_J}{\rho_J} \nabla w_J(\boldsymbol{x}) \cdot \boldsymbol{\sigma}_J$$

$$\dot{\boldsymbol{v}}_I = \frac{1}{\rho_I} \sum_{J=1}^{N} \frac{m_J}{\rho_J} \nabla w_{IJ} \cdot \boldsymbol{\sigma}_J$$

速度梯度张量的核函数近似

$$\nabla \boldsymbol{v}_I = \sum_{J=1}^{N} \frac{m_J}{\rho_J} \boldsymbol{v}_J \nabla w_{IJ}$$

- 显式积分
- 人工体积粘性

基于局部弱形式和边界积分方程的无网格法

- MLPG
  - MLS
  - LPG
- LBIE
  - MLS
  - LBIE
- BNM
  - MLS
  - BIE

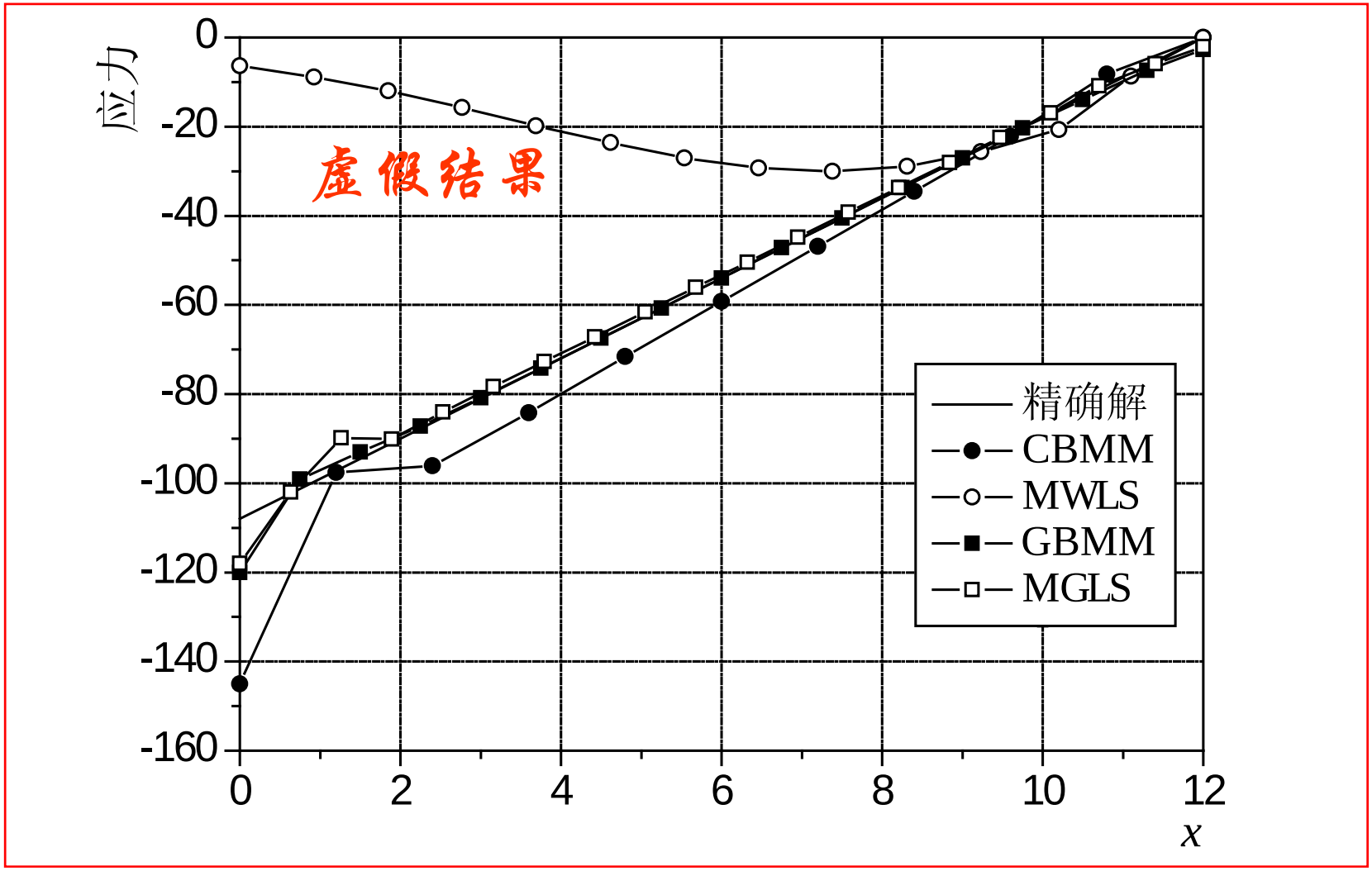
加权最小二乘型无网格法

$$\Pi = \int_\Omega (\sigma_{ij,j} + \bar{f}_i)(\sigma_{ik,k} + \bar{f}_i)\mathrm{d}\Omega$$
$$+\int_{\Gamma_u} \lambda_u (u_i - \bar{u}_i)(u_i - \bar{u}_i)\mathrm{d}\Gamma + \int_{\Gamma_t} \lambda_t (\sigma_{ij} n_j - \bar{t}_i)(\sigma_{ik} n_k - \bar{t}_i)\mathrm{d}\Gamma$$

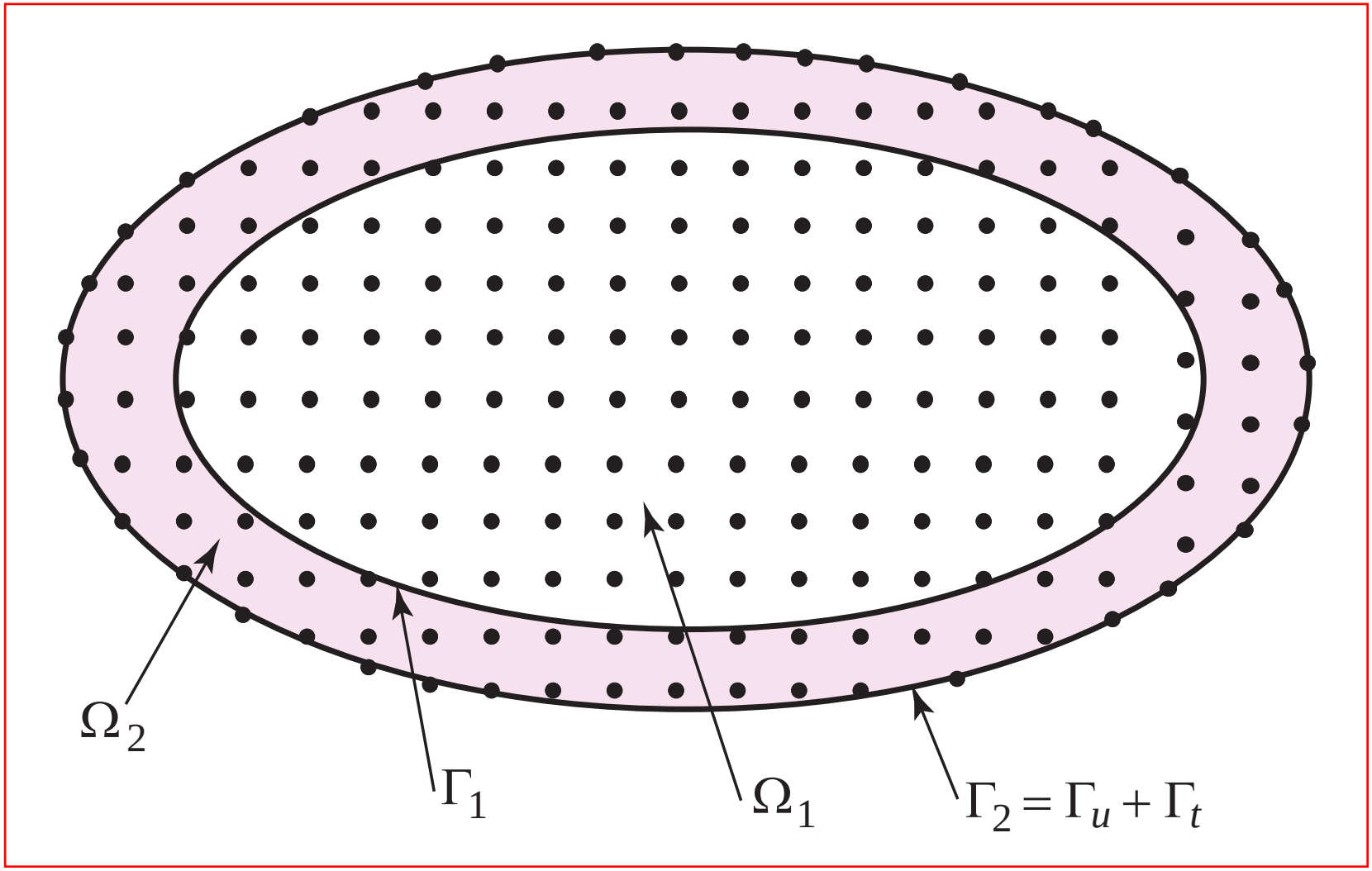
$$\delta\Pi = \int_\Omega \delta\sigma_{ik,k}(\sigma_{ij,j} + \bar{f}_i)\mathrm{d}\Omega$$
$$+\int_{\Gamma_u} \lambda_u \delta u_i (u_i - \bar{u}_i)\mathrm{d}\Gamma + \int_{\Gamma_t} \lambda_t \delta\sigma_{ik} n_k (\sigma_{ij} n_j - \bar{t}_i)\mathrm{d}\Gamma = 0$$

$$\delta\Pi = \sum_{s=1}^{N} \delta\sigma_{ik,k}(\sigma_{ij,j} + \bar{f}_i)\Big|_{\boldsymbol{x}=\boldsymbol{x}_s}$$
$$+\sum_{s=1}^{N_u} \lambda_u \delta u_i (u_i - \bar{u}_i)\Big|_{\boldsymbol{x}=\boldsymbol{x}_s} + \sum_{s=1}^{N_t} \lambda_t \delta\sigma_{ik} n_k (\sigma_{ij} n_j - \bar{t}_i)\Big|_{\boldsymbol{x}=\boldsymbol{x}_s}$$

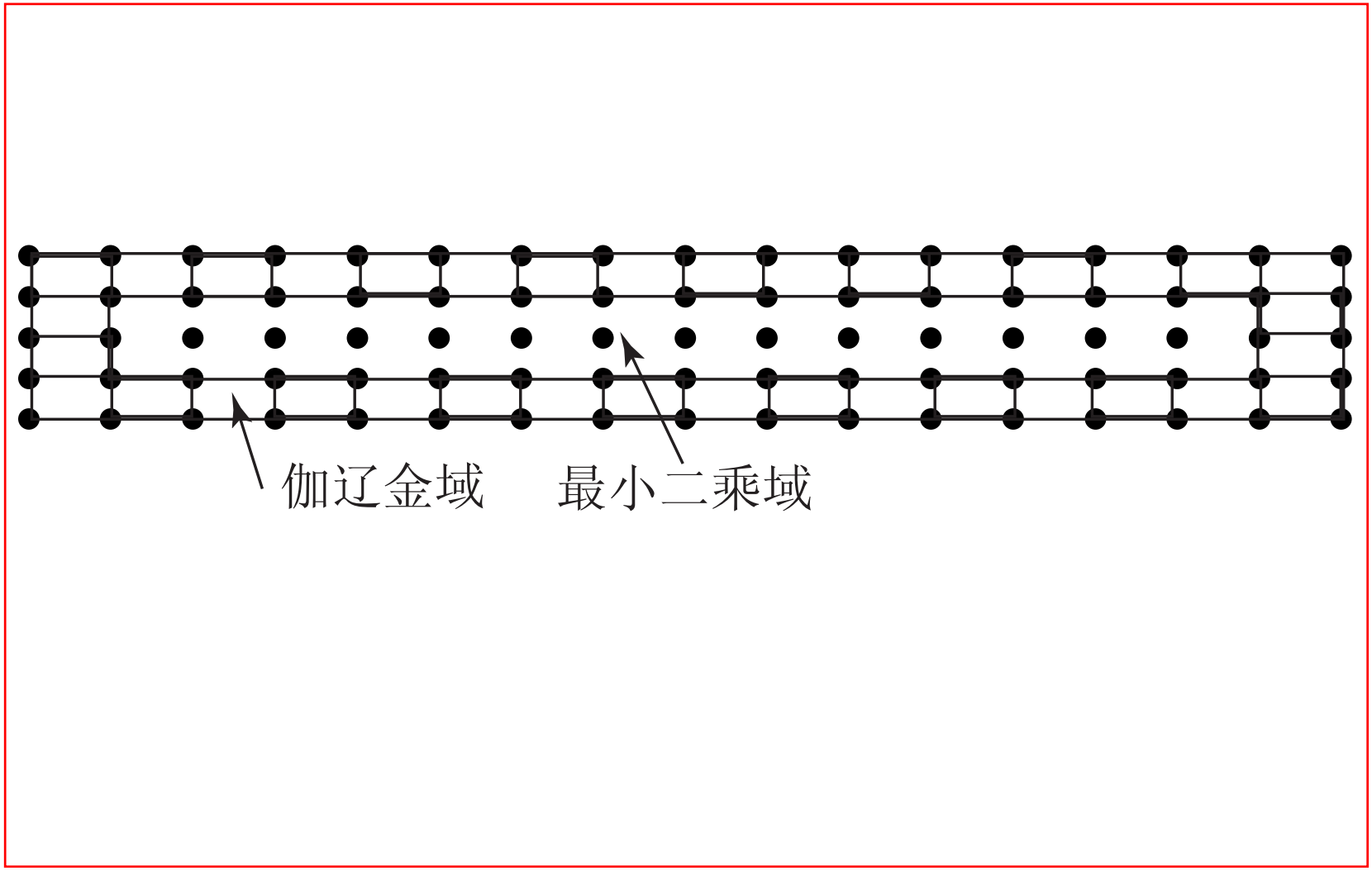
加权最小二乘型无网格法



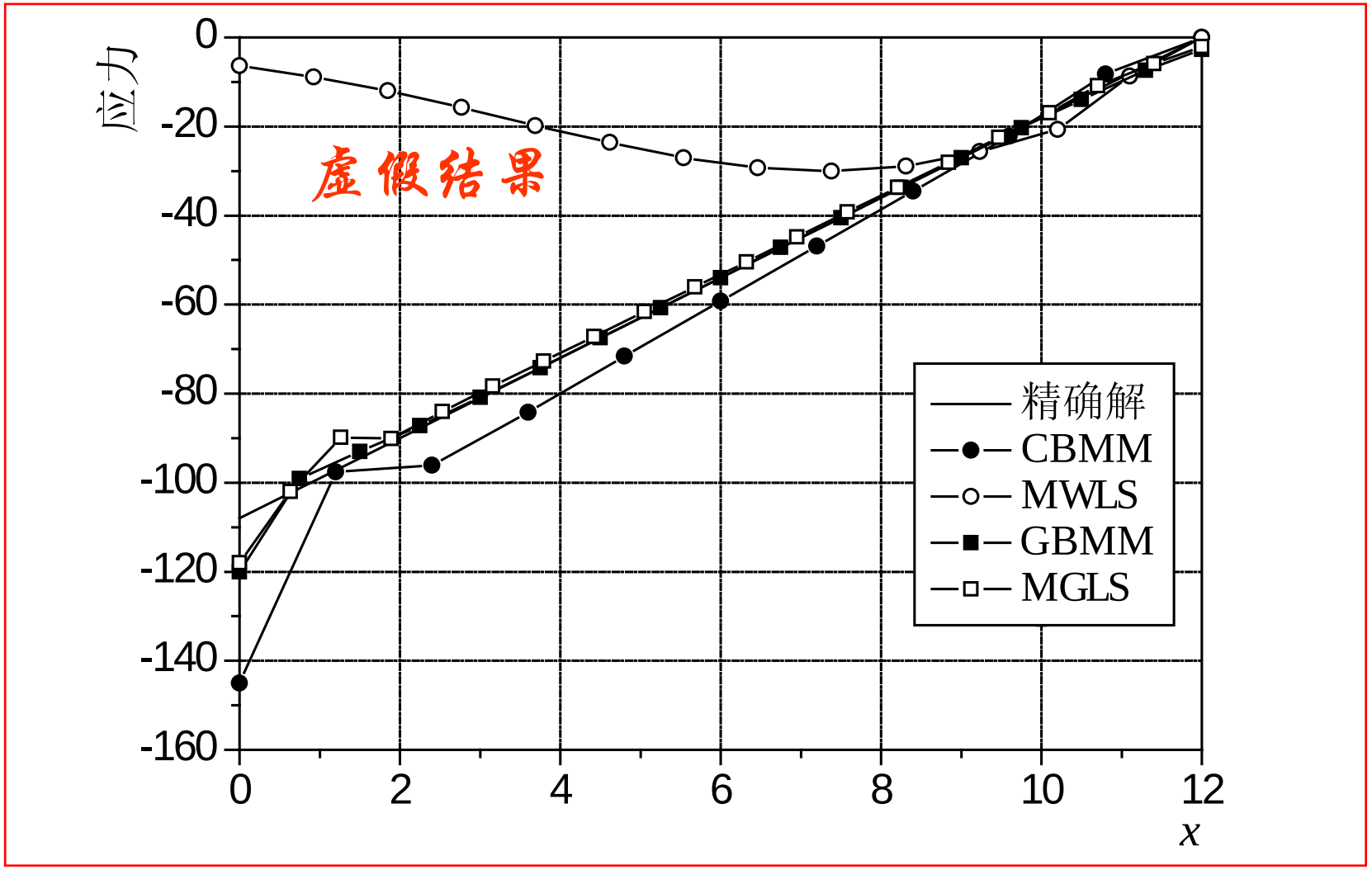
Galerkin Least Square



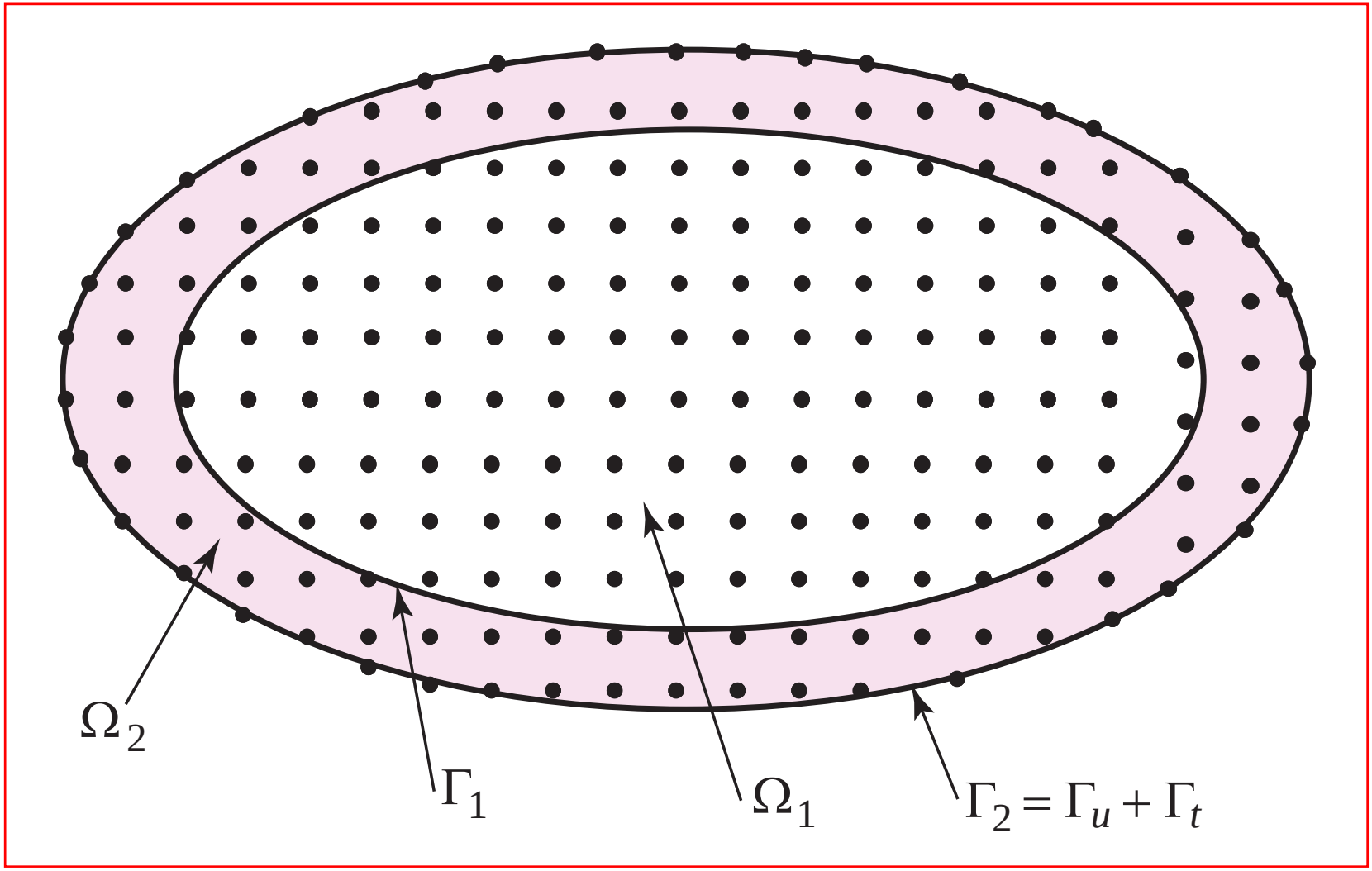
Galerkin Least Square



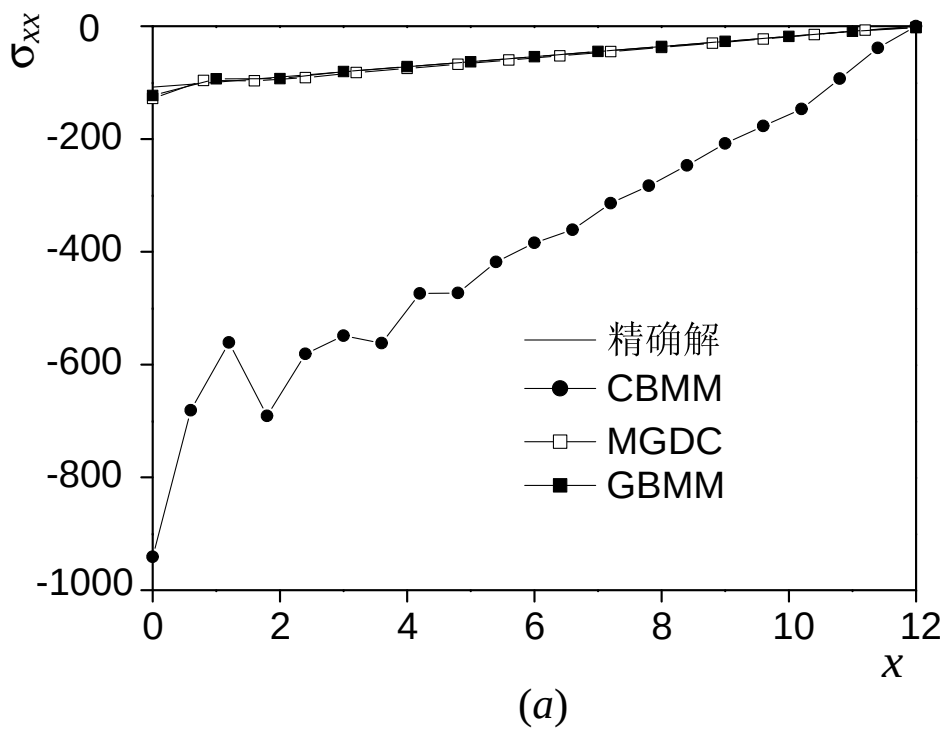
Galerkin Least Square



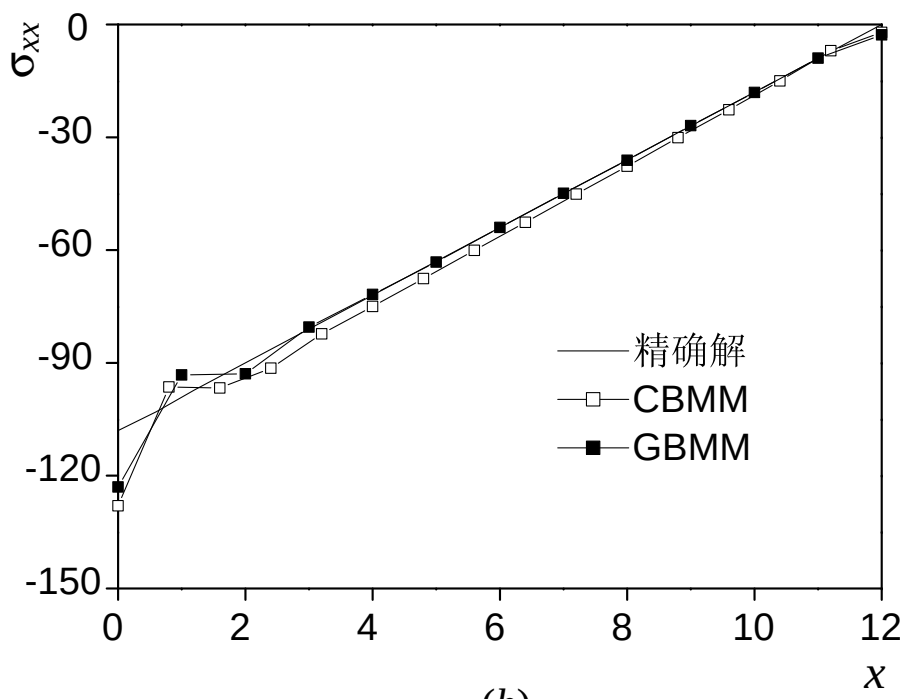
Galerkin Collocation



Galerkin Collocation



(a)

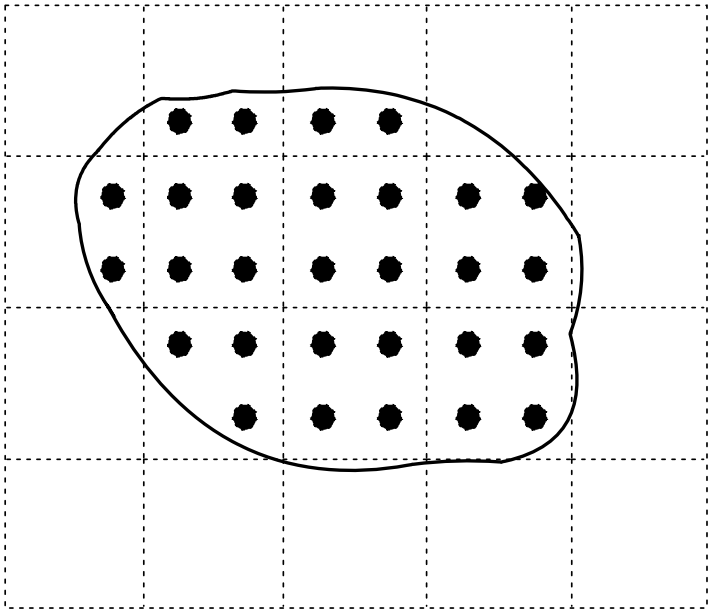


(b)

物质点法 (MPM)

- Particle in Cell (PIC)，1955
- Material Point Method (MPM)，Sulsky, 1994

  连续体离散成一组带有质量的质点，质点携带了所有物质信息，其运动代表物质的变形。



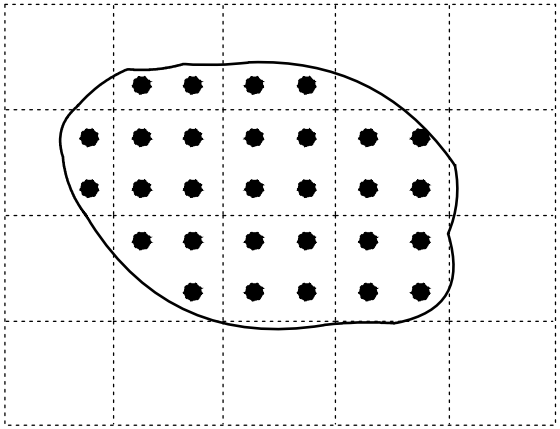
$$\rho(\boldsymbol{x}) = \sum_{p=1}^{n_p} m_p \delta(\boldsymbol{x} - \boldsymbol{x}_p)$$

$$x_{ip} = \sum_{I=1}^{n_g} N_{Ip} x_{iI}$$

$$N_{Ip} = N_I(\boldsymbol{x}_p)$$

物质点法 — 基本思想

- 每一时间步：
  - 将物质信息映射到背景网格结点处，并在背景网格上建立并求解动量方程
  - 求得的结果再映射回物质点，得到下一时刻物质点所携带的物质信息
  - 丢弃变形后的背景网格，重新构造背景网格
- 物质点固连在背景网格上 — 更新拉格朗日格式
- 结合Lagrange方法和Euler方法的优点



物质点法 — 离散格式

- 动量方程的等效积分弱形式：

$$\delta \dot{w} = \delta \dot{w}^{\text{int}} - \delta \dot{w}^{\text{ext}} + \delta \dot{w}^{\text{kin}} = 0$$

$$\delta \dot{w}^{\text{int}} = \int_V \delta v_{i,j} \sigma_{ji} \mathrm{d}V$$

$$\delta \dot{w}^{\text{ext}} = \int_V \delta v_i \rho b_i \mathrm{d}V + \int_{A_t} \delta v_i \bar{t}_i \mathrm{d}A$$

$$\delta \dot{w}^{\text{kin}} = \int_V \delta v_i \rho \ddot{u}_i \mathrm{d}V$$

物质点法 — 离散格式

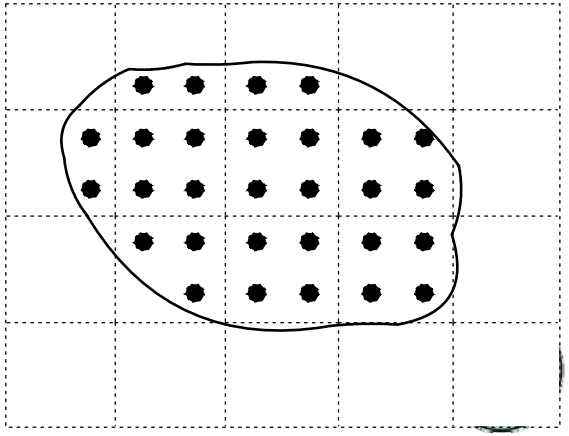
$$\delta \dot{w}^{\text{kin}} = \int_V \delta v_i \rho \ddot{u}_i \mathrm{d}V \Leftarrow \rho(\boldsymbol{x}) = \sum_{p=1}^{n_p} m_p \delta(\boldsymbol{x} - \boldsymbol{x}_p)$$

$$= \sum_{p=1}^{n_p} \delta v_{ip} m_p \ddot{u}_{ip} \Leftarrow \ddot{u}_{ip} = \sum_{J=1}^{n_g} N_{Jp} \ddot{u}_{iJ} \qquad \delta v_{ip} = \sum_{I=1}^{n_g} N_{Ip} \delta v_{iI}$$

$$= \sum_{I=1}^{n_g} \sum_{J=1}^{n_g} \sum_{p=1}^{n_p} \delta v_{iI} N_{Ip} m_p N_{Jp} \ddot{u}_{iJ} \Leftarrow m_{IJ} = \sum_{p=1}^{n_p} m_p N_{Ip} N_{Jp}$$

$$= \sum_{I=1}^{n_g} \sum_{J=1}^{n_g} \delta v_{iI} m_{IJ} \ddot{u}_{iJ} \Leftarrow p_{iI} = \sum_{J=1}^{n_g} m_{IJ} v_{iJ} = \sum_{p=1}^{n_p} m_p v_{ip} N_{Ip}$$

$$= \sum_{I=1}^{n_g} \delta v_{iI} \dot{p}_{iI}$$



物质点法 — 离散格式

$$\delta \dot{w}^{\text{int}} = \int_V \delta v_{i,j} \sigma_{ji} \mathrm{d}V \quad \Leftarrow \quad \sigma^*_{ji} = \sigma_{ji} / \rho$$

$$= \int_V \delta v_{i,j} \sigma^*_{ji} \rho \mathrm{d}V \quad \Leftarrow \quad \rho(\boldsymbol{x}) = \sum_{p=1}^{n_p} m_p \delta(\boldsymbol{x} - \boldsymbol{x}_p)$$

$$= \sum_{p=1}^{n_p} m_p \delta v_{ip,j} \sigma^*_{jip} \quad \Leftarrow \quad \delta v_{ip} = \sum_{I=1}^{n_g} N_{Ip} \delta v_{iI}$$

$$= \sum_{I=1}^{n_g} \sum_{p=1}^{n_p} \delta v_{iI} \sigma_{jip} N_{Ip,j} \frac{m_p}{\rho_p} \quad \Leftarrow \quad f^{\text{int}}_{iI} = \sum_{p=1}^{n_p} \sigma_{jip} N_{Ip,j} \frac{m_p}{\rho_p}$$

$$= \sum_{I=1}^{n_g} \delta v_{iI} f^{\text{int}}_{iI}$$

物质点法 — 离散格式

$$\delta \dot{w}^{\text{ext}} = \int_V \delta v_i \rho b_i \mathrm{d}V + \int_{A_t} \delta v_i \bar{t}_i \mathrm{d}A \Leftarrow \rho(\boldsymbol{x}) = \sum_{p=1}^{n_p} m_p \delta(\boldsymbol{x} - \boldsymbol{x}_p)$$

$$= \sum_{p=1}^{n_p} m_p \delta v_{ip} b_{ip} + \int_{A_t} \delta v_i \bar{t}_i \mathrm{d}A \Leftarrow \delta v_{ip} = \sum_{I=1}^{n_g} N_{Ip} \delta v_{iI}$$

$$= \sum_{I=1}^{n_g} \sum_{p=1}^{n_p} \delta v_{iI} N_{Ip} m_p b_{ip} + \qquad \delta v_i = \sum_{I=1}^{n_g} N_I \delta v_{iI}$$

$$\sum_{I=1}^{n_g} \delta v_{iI} \int_{A_t} N_I \bar{t}_i \mathrm{d}A \quad \Leftarrow \quad f_{iI}^{\text{ext}} = \sum_{p=1}^{n_p} N_{Ip} m_p b_{ip} + \int_{A_t} N_I \bar{t}_i \mathrm{d}A$$

$$= \sum_{I=1}^{n_g} \delta v_{iI} f_{iI}^{\text{ext}}$$

物质点法 — 离散格式

$$\delta \dot{w} = \delta \dot{w}^{\text{int}} - \delta \dot{w}^{\text{ext}} + \delta \dot{w}^{\text{kin}} = 0 \quad \Rightarrow \quad \dot{p}_{iI} = f_{iI}$$

$$\delta \dot{w}^{\text{kin}} = \sum_{I=1}^{n_g} \delta v_{iI} \dot{p}_{iI} \qquad f_{iI} = f_{iI}^{\text{ext}} - f_{iI}^{\text{int}}$$

$$\delta \dot{w}^{\text{int}} = \sum_{I=1}^{n_g} \delta v_{iI} f_{iI}^{\text{int}} \qquad f_{iI}^{\text{int}} = \sum_{p=1}^{n_p} \sigma_{jip} N_{Ip,j} \frac{m_p}{\rho_p}$$

$$\delta \dot{w}^{\text{ext}} = \sum_{I=1}^{n_g} \delta v_{iI} f_{iI}^{\text{ext}} \qquad f_{iI}^{\text{ext}} = \sum_{p=1}^{n_p} N_{Ip} m_p b_{ip} + \int_{A_t} N_I \bar{t}_i \mathrm{d}A$$

物质点法 — 离散格式

$$p_{iI} = \sum_{J=1}^{n_g} m_{IJ} v_{iJ} \qquad m_{IJ} = \sum_{p=1}^{n_p} m_p N_{Ip} N_{Jp}$$

$p_{iI} \xrightarrow{?} v_{iJ}$ 质量阵求逆！

对角质量阵 $m_I = \sum_{J=1}^{n_g} m_{IJ} = \sum_{p=1}^{n_p} m_p N_{Ip}$

$$p_{iI} = m_I v_{iI}$$

物质点法 — 数值算法步骤

- 已知t^k时刻的物理量，求t^{k+1}时刻的物理量

1.更新网格结点数据

$$m_I^k = \sum_p m_p N_{Ip}^k$$

$$p_{iI}^k = \sum_{p=1}^{n_p} m_p v_{ip}^k N_{Ip}^k$$

$$f_{iI}^k = f_{iI}^{\text{int},k} - f_{iI}^{\text{ext},k}$$

2.在背景网格结点上积分动量方程并施加固定边界条件

$$p_{iI}^{k+1} = p_{iI}^k + f_{iI}^k \Delta t$$

在固定边界上 $p_{iI}^{k+1} = 0,\ f_{iI}^k = 0$

物质点法 — 数值算法步骤

3.更新质点位置和速度

$$\bar{v}_{ip}^{k+1} = \sum_{I=1}^{8} \frac{p_I^{k+1}}{m_I^k} N_{Ip}^k$$

$$a_{ip}^k = \sum_{I=1}^{8} \frac{f_I^k}{m_I^k} N_{Ip}^k$$

$$x_{ip}^{k+1} = x_{ip}^k + \bar{v}_{ip}^{k+1} \Delta t$$

$$v_{ip}^{k+1} = v_{ip}^k + a_{ip}^k \Delta t$$

物质点法 — 数值算法步骤

4.将速度映射回背景网格结点

$$v_{iI}^{k+1} = \frac{\sum_{p} m_p v_p^{k+1} N_{Ip}^k}{m_I^k}$$

5.计算应变增量和旋率增量

$$\Delta\varepsilon_{ijp}^k = \frac{1}{2}\sum_{I=1}^{8}[N_{Ip,j}^{k+1} v_{iI}^{k+1} + N_{Ip,i}^{k+1} v_{jI}^{k+1}]\Delta t$$

$$\Delta\Omega_{ijp}^k = \frac{1}{2}\sum_{I=1}^{8}[N_{Ip,j}^{k+1} v_{iI}^{k+1} - N_{Ip,i}^{k+1} v_{jI}^{k+1}]\Delta t$$

物质点法 — 数值算法步骤

6.更新密度，应力

$$\rho_p^{k+1} = \rho_p^k /(1+\Delta\varepsilon_{iip}^k)$$

$$\sigma_{ijp}^{k+1} = \sigma_{ijp}^k + \Delta r_{ijp}^k + \Delta\sigma_{ijp}^k$$

其中 $\Delta r_{ijp}^k = \Delta\Omega_{ijp}^k \sigma_{ijp}^k - \sigma_{ijp}^k \Delta\Omega_{ijp}^k$

$\Delta\sigma_{ijp}^k$根据弹塑性本构关系更新

7. 进行下一个时间步循环